CHAPTER 5

The Dirac equation on the light cone
Majorana electrons and magnetic monopoles

1. Introduction. How the Majorana field appears in the theory of a magnetic monopole.

In the first chapters, we have developed the theory of a massless linear monopole, the quantized magnetic charge of which generalizes the Dirac formula. The neutrino appears as the fundamental zero state of the magnetic charge. The monopole is massless, because the linear Dirac mass term would violate the *chiral gauge* invariance : $\Psi \to \exp\left(i \frac{g}{\hbar c} \gamma_5 \phi\right) \Psi$, which ensures the conservation of magnetism.

Nevertheless, in the Chapter 4, we gave a generalization (4.2) of the linear equation (2.16), owing to the introduction of a *nonlinear mass term,* invariant with respect to the chiral gauge. There is an infinite family of such mass terms depending on an arbitrary function of a *chiral invariant* which is equal (up to a constant factor) to the space curvature.

Now, we shall reexamine the problem of mass in another way. We shall consider *the Dirac equation on the relativistic light-cone*, which gives a generalization of the *Majorana condition.* This result was achieved in (*Lochak 7, 8,15*). The main idea is that the *Majorana condition*, which reduces the Dirac equation to an "abbreviated" form, will be substituted by the condition that *the chiral invariant equals zero*, which is equivalent to write **the Dirac equation on the relativistic light-cone**, **if we define the light-cone by the condition that the electric current (i.e. the velocity of the particle) is isotropic** :

$$J_\mu J_\mu = 0 \tag{5.1}$$

But, in virtue of the algebraic relations (2.24) :

$$-J_\mu J_\mu = \Sigma_\mu \Sigma_\mu = \omega_1^2 + \omega_2^2 \left[= 4\left(\xi^+\eta\right)\left(\eta^+\xi\right) \text{in the Weyl representation} \right]$$

Thus, the condition (5.1) means that the chiral invariant equals zero on the light-cone :

$$\rho = \sqrt{\omega_1^2 + \omega_2^2} = 0 \tag{5.2}$$

It must be noticed that this definition is compatible with the conservation of electricity and magnetism because ρ is invariant under the ordinary gauge and under the chiral gauge. Let us now consider the equations of the magnetic monopole (Ch. 4), with a nonlinear mass term.

So w have, in the Dirac representation :

$$\gamma_\mu \left(\partial_\mu - \frac{g}{\hbar c} \gamma_5 G_\mu\right) \Psi + \frac{1}{2} \frac{m(\rho)\, c}{\hbar} \left(\omega_1 - i\,\gamma_5 \omega_2\right) = 0 \tag{5.3}$$

And then in the Weyl representation :

$$
\begin{aligned}
&\frac{1}{c}\,\partial_t\xi - \vec{s}.\nabla\xi - i\,\frac{g}{\hbar c}\left(W + \vec{s}.\vec{B}\right)\xi + i\,\frac{m\left(2\left|\xi^+\eta\right|\right)c}{\hbar}\left(\eta^+\xi\right)\eta = 0 \\
&\frac{1}{c}\,\partial_t\eta + \vec{s}.\nabla\eta + i\,\frac{g}{\hbar c}\left(W - \vec{s}.\vec{B}\right)\eta + i\,\frac{m\left(2\left|\xi^+\eta\right|\right)c}{\hbar}\left(\xi^+\eta\right)\xi = 0
\end{aligned}
\qquad \left\{B_\mu = \left(-i\,\mathbf{G},\, W\right)\right\} \qquad (5.4)
$$

These equations are invariant with respect to the chiral gauge transformation and they represent a magnetic monopole. It was shown (Ch. 4) that the solutions of equations as (5.3) and (5.4) are divided into : bradyon states (slower than light), tachyon states (faster than light) and luxon states (with the light velocity).

Just like the linear equations of the monopole these **nonlinear equations admit a « nonlinear neutrino »** as a particular case for a zero charge : $g = 0$, which means that such a nonlinear neutrino must have the same three states as the nonlinear monopole : bradyon, tachyon and luxon. This hypothesis was precedingly formulated in another frame, by Mignani and Recami (*Mignani* & *Recami* and *Recami* & *Mignani*).

Now the luxon state corresponds to the cancellation of the mass terms in the equations (5.3), (5.4), which are thus reduced to the linear equations (2.16), (3.4). But here, it does not mean a simple elimination of the mass term by the annihilation of a mass coefficient, because *m* is not a simple coefficient, but a function. So that it means **a nonlinear condition on the wave functions** :

$$
\rho = 0 \quad \Rightarrow \quad \omega_1 = \omega_2 = 0 \quad \Rightarrow \quad \xi^+\eta = 0 \qquad (5.5)
$$

The cancellation of the nonlinear term under the condition (5.5) does not imply the cancellation of the wave. The condition (5,5) is not exactly equivalent to the Majorana condition (*Majorana* and *Mac Lennan*) which reads : $\psi = \psi_c \ \left(\psi_c = \psi_{charge\ conjugated}\right)$, it gives a slightly more general condition (*Lochak* 5) :

$$
\psi = e^{2i\frac{e}{\hbar c}\theta}\gamma_2\psi^* = e^{2i\frac{e}{\hbar c}\theta}\psi_c \quad \Rightarrow \quad \xi = e^{2i\frac{e}{\hbar c}\theta} i s_2\eta^*, \quad \eta = -e^{2i\frac{e}{\hbar c}\theta} i s_2\xi^* \qquad (5.6)
$$

where $\theta(x,t)$ is an arbitrary phase (the coefficient $2e/\hbar c$ will be useful later).

In other words, the ψ state definite by (5.6) is its own charge-conjugated, but up to an arbitrary phase : this is almost the Majorana condition, which gives not exactly the Majorana - « abbreviated equation ». Later, we shall consider an equation, which will not be « abbreviated », from the linear Dirac equation of the electron, but from the nonlinear equation of the monopole.

The fact that such a condition arises from the monopole theory, leads us to explore it more precisely. Since the abbreviated Majorana equation was already suggested as a possible equation for the neutrino ; and so, we can ask : why not for a magnetic monopole ?

Nevertheless, we shall consider not the magnetic case at first, but the electric one. And we wish to warn the reader against a possible disappointment because the electric case will be longer examined than the magnetic one, which is paradoxical in the present book. The reason is that, the magnetic case is far much complicated than the electric one, and that the last is interesting in itself. And it is not so elementary ! And it is illuminating for our subject.

2) The electric case : Lagrangian representation and gauge invariance of the Majorana field.

Several authors (*Mac Lennan,Case, Berzeszki*) alluded to the problem of a lagrangian representation of the Majorana field and they admitted that such a representation is impossible. We shall see that it is wrong, but it is interesting to see where the difficulty is.

Using (2.11) and the Majorana condition : $\psi = \psi_c$, for an electrically charged particle in the presence of an electomagnetic field, the Majorana equation may be written as :

$$\gamma_\mu(\partial_\mu - \frac{ie}{\hbar c}A_\mu)\psi + \frac{m_0 c}{\hbar}\psi_c = 0 \qquad (5.7)$$

If we directly try to find a lagrangian, for such an equation it must contain a term as :

$$\bar{\psi}\psi_c = \psi^+\gamma_4\gamma_2\psi^* \qquad (5.8)$$

But we have, on the other side :

$$\gamma_k = i\alpha_4\alpha_k \ (k = 1,2,3),\ \gamma_4 = \alpha_4$$
$$\alpha_k = \begin{bmatrix} 0 & s_k \\ s_k & 0 \end{bmatrix}, \alpha_4 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix};\ (s_k = \text{Pauli matrices}) \qquad (5.9)$$

Introducing these expressions in (5.8) we have identically : $\bar{\psi}\psi_c = 0$ and the corresponding term disappears from the lagrangian : here is the difficulty. But we shall proceed in another way : **we consider the Majorana field as a constrained state of the Dirac field** and we express this constraint under the form (5.5). Thus we define the « Majorana lagrangian » as a Dirac lagrangien L_D to which we add a constraint term with a Lagrange parameter λ :

$$L_M = L_D + \frac{\lambda}{2}(\omega_1^2 + \omega_2^2) \qquad (5.10)$$

ω_1 and ω_2 are taken in (2.18), so that the variation of L_M, with respect to ψ, gives :

$$\gamma_\mu(\partial_\mu - \frac{ie}{\hbar c}A_\mu)\psi + \frac{m_0 c}{\hbar}\psi + \lambda(\omega_1 - i\omega_2\gamma_5)\psi = 0 \qquad (5.11)$$

This equation looks like our nonlinear equation (see *Ch. 4* and (*Lochak 4, 7*)), but here, we have a mass term and an electric potential instead of the magnetic potential. In this form, the equation was found by HermannWeyl (see *Weyl*) and afterwards, rediscovered later by other authors. The aim of Weyl (related to the general relativity) was quite different from ours.

Now, we vary the Lagrangian L_M (5.10) with respect to λ which gives, using (5.6) :

$$\gamma_\mu(\partial_\mu - \frac{ie}{\hbar c}A_\mu)\psi + \frac{m_0 c}{\hbar} e^{2i\frac{e}{\hbar c}\theta}\psi_c = 0 \tag{5.12}$$

It is the Majorana equation (5.7) with an *arbitrary* phase θ. We could write at once $\theta = 0$ in order to find (5.7), but it would be a bad idea because this phase is important : owing to this phase, the equation (5.12) is *gauge invariant,* while the equation (5.7) is not. By the way, nobody worried about that !

In the present case, the gauge invariance of (5.12) is a trivial consequence of the invariance of the lagrangian (5.10). But the invariance of the equation (5.12) can be also directly directly demonstrated after the transformation :

$$\psi \to e^{i\frac{e}{\hbar c}\varphi}\psi,\ \ A_\mu \to A_\mu - \partial_\mu\varphi,\ \ \theta \to \theta + \varphi \tag{5.13}$$

Now – and only now - the phase θ may be absorbed in the gauge and disappear. Therefore, we must chose at first choose the gauge and **only then** cancel θ to find the equation (5.7).

Therefore, **the Majorana equation cannot be considered as an independent equation : it is only the equation of a particular state (definined by a Lagrange multiplier) of the Dirac equation of the electron**. And it is not gauge invariant : only (5.11) is invariant. Nevertheless we shall see that the Majorana equation may be considered in itself, but this second interpretation is not equivalent to the preceding one.

3. Two-component electric equations. Symmetry and conservation laws.

Now, owing to (3.4), we find the Weyl representation of the class of the just definite solutions of the Dirac equation :

$$(\pi_0 + \vec{\pi}.\mathbf{s})\xi - i\, m_0 c\, e^{\frac{2ie}{\hbar c}\theta} s_2\xi^* = 0 \tag{5.14$_1$}$$

$$(\pi_0 - \vec{\pi}.\mathbf{s})\eta + i\, m_0 c\, e^{2i\frac{e}{\hbar c}\theta} s_2\eta^* = 0 \tag{5.14$_2$}$$

$$\pi_0 = \frac{1}{c}(i\hbar\frac{\partial}{\partial t} + eV),\ \ \pi = (-i\hbar\nabla + \frac{e}{c}\mathbf{A}),\ \ \left\{A_\mu = (\mathbf{A},\ iV)\right\} \tag{5.15}$$

The formulae (5.14), are manifestly C, P and T invariant, but it is interesting to verify the property directly. Elementary calculations show, indeed, that the system (5.14) remains invariant by the following transformations using the Curie laws or those deduced from them [see : *Ch. 4* and (*Poincaré*)] :

$$\begin{aligned}
&(C) : i \to -i,\quad e \to -e,\quad \xi \to e^{2i\frac{e}{\hbar c}\theta} i\, s_2\eta^*,\quad \eta \to -e^{2i\frac{e}{\hbar c}\theta} i\, s_2\xi^* \\
&(P) : \mathbf{x} \to -\mathbf{x},\quad \mathbf{A} \to -\mathbf{A},\quad \xi \to i\eta \quad \eta \to -i\xi \\
&(T) : e \to -e, t \to -t, V \to -V, \eta \to s_2\xi^*, \xi \to -s_2\eta^*
\end{aligned} \tag{5.16}$$

The P transformation can be written in another way :

$$(P) : \mathbf{x} \to -\mathbf{x},\quad \mathbf{A} \to -\mathbf{A},\quad \xi \longleftrightarrow \eta,\quad \theta \to \theta + \frac{\pi}{2}\frac{\hbar c}{e} \tag{5.17}$$

And the gauge transformation takes the form :

$$\xi \to e^{i\frac{e}{\hbar c}\varphi}\xi, \quad \eta \to e^{i\frac{e}{\hbar c}\varphi}\eta, \quad \mathbf{A} \to \mathbf{A} - \nabla\varphi, \; V \to V + \frac{1}{c}\frac{\partial\varphi}{\partial t}, \; \theta \to \theta + \varphi \tag{5.18}$$

It can be verified that the system (5.14) remains invariant under (5.18), **which entails the conservation (3.7) of the chiral currents.** It is important to note this conservation because it is true for a magnetic monopole (*Ch. 3*), and here we see that it is also true for the solutions of the Dirac equation in the abbreviated case of an electron, **restricted by the constraint (5.5) :** $\rho = \sqrt{\omega_1^2 + \omega_2^2} = 0$. But this splitting into two equations is not true in the general case of the Dirac equation, which conserves only the electric current (the sum of the chiral currents) but not the magnetic current (*Ch. 3*).

In the abbreviated electric case, the electric current is isotropic, the solutions of the Dirac equation are on the light cone, and the magnetic current disappears. Given that the equations (5.14_1) and (5.14_2) are splitted by the condition $\xi^+\eta = 0$ we can restrict ouselves, to only one of them - say (5.14_1) - and consider it in itself. **This restricted equation is a chiral state of the electron. The equation (5.14_2) is the chiral conjugate of (5.14_1), which means, owing to (5.16), that the image in a mirror is the time inverse of (5.14_1).**

4. The chiral state of the electron in an electric coulomb field.

Majorana considered that the equality $\psi = \psi_c$, introduced in the Dirac equation gives something like a joint theory of the electron and the positron. But it is not so, because the preceeding equality is only a constraint imposed to the electron. Nevertheless we shall find a hybrid-state : a kind of mixture of the electron and the positron. To show this, we shall solve the equation (5.14_1) in an electric coulomb field by introducing the expressions :

$$eV = \frac{-e^2}{r}, \; \mathbf{A} = 0, \; \theta = \frac{\pi}{4}\frac{\hbar c}{e} \tag{5.19}$$

It gives the equation :

$$\left[\frac{1}{c}\left(i\hbar\frac{\partial}{\partial t} - \frac{e^2}{r}\right) - i\hbar\mathbf{s}.\nabla\right]\xi + m_0 c s_2 \xi^* = 0 \tag{5.20}$$

The difficulty lies obviously in the complex conjugated ξ^*. So, let us introduce the spherical functions with spin [see *Kramers, Bohm, Akhieser & Berestetski*] :

$$\Omega_\ell^m(+) = \begin{bmatrix} \left(\frac{\ell + m}{2\ell + 1}\right)^{\frac{1}{2}} Y_\ell^{m-1} \\ \left(\frac{\ell - m + 1}{2\ell + 1}\right)^{\frac{1}{2}} Y_\ell^m \end{bmatrix}, \quad \Omega_\ell^m(-) = \begin{bmatrix} \left(\frac{\ell - m + 1}{2\ell + 1}\right)^{\frac{1}{2}} Y_\ell^{m-1} \\ -\left(\frac{\ell + m}{2\ell + 1}\right)^{\frac{1}{2}} Y_\ell^m \end{bmatrix} \tag{5.21}$$

Y_ℓ^m are the Laplace spherical functions (ℓ=0, 1, 2, . . . ; $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$) :

$$Y_l^m(\theta,\varphi) = \frac{(-1)^m}{2^l l!}\left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}}\left(\frac{(l+m)!}{(l-m)!}\right)^{\frac{1}{2}}\frac{e^{im\varphi}}{\sin^l\theta}\frac{d^{l-m}}{d\theta^{l-m}}sin^{2l}\theta \tag{5.22}$$

Now, we have the following equalities (Appendix A) :

$$\mathbf{s.n}\,\Omega_{l-1}^{m}(+)=\Omega_{l}^{m}(-);\quad \mathbf{s.n}\,\Omega_{l}^{m}(-)=\Omega_{l-1}^{m}(+)$$
$$\mathbf{s.n}\,s_2\,\Omega*_{l-1}^{m}(+)=i(-1)^{m+1}\,\Omega_{l}^{-m+1}(-) \tag{5.23}$$
$$\mathbf{s.n}\,s_2\,\Omega*_{l}^{m}(-)=i(-1)^{m}\,\Omega_{l-1}^{-m+1}(+)$$

$$\mathbf{n}=\frac{\mathbf{r}}{r};x=r\cos\varphi\sin\theta,\ y=r\sin\varphi\sin\theta,\ z=r\cos\theta\ \vec{n}=\frac{\vec{r}}{r};x=r\cos\varphi\sin\theta,\ y=r\sin\varphi\sin\theta,\ z=r\cos\theta \tag{5.24}$$

And we look for a solution of (5.20) of the form :

$$\xi=\sum_{m}F_{\ell-1}^{m}(t,r)\Omega_{\ell-1}^{m}(+)+\sum_{m'}G_{\ell}^{m'}(t,r)\Omega_{\ell}^{m'}(-) \tag{5.25}$$

But it is impossible to immediately separate the variables t and r. It is only possible to separate the angular variables φ and θ. Following a classical procedure in the Dirac theory (*Kramers, Akhieser & Berestetski*) we introduce (5.25) in (5.20) multiplying on the left by **s.n** . Owing to (5.23) we find :

$$\begin{aligned}&\frac{1}{c}(i\hbar\frac{\partial}{\partial t}-\frac{e^2}{r})\left[F_{\ell-1}^{m}\Omega_{\ell}^{m}(-)+\sum_{m'}G_{\ell}^{m'}\Omega_{\ell-1}^{m'}(+)\right]\\&=i\hbar\mathbf{s.n}\,\mathbf{s}.\nabla\left[\sum_{m}F_{\ell-1}^{m}\Omega_{\ell-1}^{m}(+)+\sum_{m'}G_{\ell}^{m'}\Omega_{\ell}^{m'}(-)\right]\\&-im_0c\left[\sum_{m}(-1)^{m+1}F_{\ell-1}^{*m}\Omega_{\ell}^{-m+1}(-)+\sum_{m'}(-1)^{m'}G_{\ell}^{*m'}\Omega_{\ell}^{-m'+1}(+)\right]\end{aligned} \tag{5.26}$$

The second member is simplified owing to the classical relations :

$$\mathbf{s.n}\ \ \mathbf{s}.\nabla=\frac{\partial}{\partial r}-\frac{1}{r}\mathbf{s}.\Lambda \tag{5.27}$$

where Λ is the orbital moment :

$$\Lambda=-i\mathbf{r}\times\nabla \tag{5.28}$$

Now, we have other relations (Appendix B) :

$$\begin{aligned}\mathbf{s}.\Lambda\Omega_{\ell-1}^{m}(+)&=(\ell-1)\,\Omega_{\ell-1}^{m}(+)\\ \mathbf{s}.\Lambda\Omega_{\ell}^{m}(-)&=-(\ell+1)\Omega_{\ell}^{m}(-)\end{aligned} \tag{5.29}$$

so that, taking into account that $\Omega_{\ell}^{m}(\pm)$ are orthonormal, we deduce from (5.26) the following system from which the angles are eliminated :

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + i\frac{\alpha}{r}\right)F^m_{\ell-1} = \left(\frac{\partial}{\partial r} + \frac{1+\ell}{r}\right)G^m_\ell + \chi(-1)^m F^{*-m+1}_{\ell-1}$$
$$\left(\frac{1}{c}\frac{\partial}{\partial t} + i\frac{\alpha}{r}\right)G^m_\ell = \left(\frac{\partial}{\partial r} + \frac{1-l}{r}\right)F^m_{\ell-1} - \chi(-1)^m G^{*-m+1}_\ell \tag{5.30}$$

$$m = -l, -l+1, \ldots, l-1, l, \quad \alpha = \frac{e^2}{\hbar c}, \quad \chi = \frac{m_0 c}{\hbar} \tag{5.31}$$

In a following step, we take the complex conjugated of (5.30), changing : $m \longleftrightarrow -m+1$:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - i\frac{\alpha}{r}\right)F^{*-m+1}_{l-1} = \left(\frac{\partial}{\partial r} + \frac{1+l}{r}\right)G^{*-m+1}_l - \chi(-1)^m F^m_{l-1}$$
$$\left(\frac{1}{c}\frac{\partial}{\partial t} - i\frac{\alpha}{r}\right)G^{*-m+1}_l = \left(\frac{\partial}{\partial r} + \frac{1-l}{r}\right)F^{*-m+1}_{l-1} + \chi(-1)^m G^m_l \tag{5.32}$$

And we combine (5.30) and (5.32), introducing the new functions :

$$\begin{aligned}
\frac{A^m_{\ell-1}(r)}{r}e^{(-1)^m i\omega t} &= F^m_{\ell-1} + (-1)^m F^{*-m+1}_{\ell-1};\\
\frac{B^m_{\ell-1}(r)}{r}e^{(-1)^m i\omega t} &= F^m_{\ell-1} - (-1)^m F^{*-m+1}_{\ell-1};\\
\frac{C^m_{\ell}(r)}{r}e^{(-1)^m i\omega t} &= G^m_{\ell} + (-1)^m G^{*-m+1}_{\ell};\\
\frac{D^m_{\ell}(r)}{r}e^{(-1)^m i\omega t} &= G^m_{\ell} - (-1)^m G^{*-m+1}_{\ell};
\end{aligned} \tag{5.33}$$

with :

$$B^m_{l-1} = (-1)^{m+1} A^{*-m+1}_{l-1}\,;\; D^m_l = (-1)^{m+1} C^{*-m+1}_l \tag{5.34}$$

With the definition (5.34), the notations (5.33) are invariant under complex conjugation and : $m \rightarrow -m+1$. Summing and subtracting (5.30) and (5.32), we find a first order system in r, [see *Ince*] :

$$r\frac{dX}{dr} = (M + Nr)X; \tag{5.35}$$

$$X = \begin{bmatrix} A^m_{\ell-1}(r) \\ B^m_{\ell-1}(r) \\ C^m_\ell(r) \\ D^m_\ell(r) \end{bmatrix}; \quad M = \begin{bmatrix} \ell & 0 & 0 & i\alpha \\ 0 & \ell & i\alpha & 0 \\ 0 & i\alpha & -\ell & 0 \\ i\alpha & 0 & 0 & -\ell \end{bmatrix}; \quad N = \begin{bmatrix} 0 & 0 & i\frac{\omega'}{c} & -\chi \\ 0 & 0 & \chi & i\frac{\omega'}{c} \\ i\frac{\omega'}{c} & \chi & 0 & 0 \\ -\chi & i\frac{\omega'}{c} & 0 & 0 \end{bmatrix}; \quad \omega' = (-1)^m \omega \tag{5.36}$$

The matrix N is diagonalized by :

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & \frac{\omega'}{\mu c} & i\frac{\chi}{\mu} \\ 0 & 1 & -i\frac{\chi}{\mu} & \frac{\omega'}{\mu c} \\ 1 & 0 & -\frac{\omega'}{\mu c} & -i\frac{\chi}{\mu} \\ 0 & 1 & i\frac{\chi}{\mu} & -\frac{\omega'}{\mu c} \end{bmatrix} \;;\; \mu = \sqrt{\frac{\omega^2}{c^2} - \chi^2} \tag{5.37}$$

Introducing the new variable :

$$Y = SX \tag{5.38}$$

the equation in (5.35) takes the form :

$$r\frac{dY}{dr} = \left\{ i\mu r\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + l\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} + \frac{\alpha}{\mu}\begin{bmatrix} \frac{\omega'}{c}s_1 + i\chi s_3 & 0 \\ 0 & -\frac{\omega'}{c}s_1 - i\chi s_3 \end{bmatrix} \right\} Y \tag{5.39}$$

μ is definite in (5.37), I is the unit matrix of the 2[nd] order and s_1, s_3 are Pauli matrices. We shall now diagonalize (5.39), changing the functions once more :

$$Z = \begin{bmatrix} V & 0 \\ 0 & s_1 V \end{bmatrix} Y \;;\; V = \left[\frac{\omega'}{2\mu c}\right]^{\frac{1}{2}} \begin{bmatrix} \left[\frac{\omega'/c}{\mu - i\chi}\right]^{\frac{1}{2}} & \left[\frac{\mu - i\chi}{\omega'/c}\right]^{\frac{1}{2}} \\ \left[\frac{\omega'/c}{\mu + i\chi}\right]^{\frac{1}{2}} & -\left[\frac{\mu + i\chi}{\omega'/c}\right]^{\frac{1}{2}} \end{bmatrix}, \tag{5.40}$$

V is chosen so that :

$$V\left(\frac{\omega'}{c}s_1 + i\chi s_3\right)V^{-1} = \mu s_3 \tag{5.41}$$

The equation takes the new form :

$$r\frac{dZ}{dr} = \left\{ i\mu r\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \ell\begin{bmatrix} 0 & s_1 \\ s_1 & 0 \end{bmatrix} + i\alpha\begin{bmatrix} s_3 & 0 \\ 0 & s_3 \end{bmatrix} \right\} Z \tag{5.42}$$

And by iteration, we find :

$$\left[r\frac{d}{dr}\right]^2 Z = \left\{ -\mu^2 r^2 + \mu r\left(i\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} - 2\alpha\begin{bmatrix} s_3 & 0 \\ 0 & s_3 \end{bmatrix} \right) + \ell^2 - \alpha^2 \right\} Z \tag{5.43}$$

All the matrices are diagonalized, and we find four independent equations for the components of Z :

$$\left[r\frac{d}{dr}\right]^2 Z_n = \left[-\mu^2 r^2 + \mu r(i\varepsilon - 2\alpha\varepsilon') + \ell^2 - \alpha^2\right] Z_n \quad (n = 1,2,3,4) \tag{5.44}$$

$$\varepsilon = 1,1,-1,-1;\ \varepsilon' = 1,-1,-1,1 \quad (\text{for } \ n = 1,2,3,4) \tag{5.45}$$

Let us put now :

$$r = \frac{i\rho}{2\mu}, W_n = \rho^{\frac{1}{2}} Z_n \tag{5.46}$$

The equation (5.44) becomes, neglecting the suffix n :

$$\frac{d^2W}{d\rho^2} + \left[-\frac{1}{4} + \frac{\frac{\varepsilon}{2} - i\alpha\varepsilon'}{\rho} + \frac{\frac{1}{4} + \alpha^2 - l^2}{\rho^2}\right] W = 0 \tag{5.47}$$

This is a Whittaker equation (*Ince, Whittaker & Watson*). The following coefficients are denoted here by *k* and *m,* keeping the classical notation for $W_{k,m}$, they are not to be confused with other precedingly used indices:

$$k = \frac{\varepsilon}{2} - i\alpha\varepsilon', \quad m = \sqrt{l^2 - \alpha^2} \tag{5.48}$$

Thus, we can take the following Whittaker functions as radial functions, provided they are square integrable at the origin :

$$W_{k,m}(\rho) = W_{\frac{\varepsilon}{2} - i\alpha\varepsilon', \sqrt{l^2 - \alpha^2}}(-2i\mu r) \tag{5.49}$$

But in the vicinity of the origin, a regular solution of (5.47) may be written in the following form (*Ince, Whittaker & Watson*), taking into account of (5.46) and (5.48) :

$$|\,W_{k,m}\,| = 2\mu r^{\frac{1}{2}+m}(1 + O(r)) \tag{5.50}$$

It must be noticed that the same coefficient m appears in all the composants W_n and thus in Z_n in (5.44) ; therefore, going up to the changes of functions (5.46), (5.40), (5.38), (5.34), (5.25), we can assert that :

$$\xi^+\xi \approx r^2(m-1) \quad (\text{ in the vicinity of } r = 0) \tag{5.51}$$

So, the value (5.48) of m shows that $\xi^+\xi$ is always integrable at the origin because $l = 0,1,2...$ But the more interesting is the behaviour at the infinity. From classical formulae, we have (*Whittaker & Watson*):

$$W_{k,m}(\rho) = e^{-\frac{1}{2}\rho}\rho^k(1 + O(\rho^{-1})) \quad \text{si} \mid \text{Arg}(-\rho) \mid < \pi \tag{5.52}$$

The condition of validity is satisfied because : $\rho = -2i\mu r$, in virtue of (5.46), so that owing to (5.48) :

$$W_{k,m}(\rho) = 2\mu r^{\frac{\varepsilon}{2}}(1 + O(r^{-1}) \qquad [\,\varepsilon = \pm 1 \text{ like in (5.44)}] \tag{5.53}$$

If we consider now the change of functions $\xi^+\xi$, we meet a little difficulty. In $r \approx 0$ we had the same exponents in (5.50) for all the components W or Z ; but now the situation is different with the exponent $\frac{\xi}{2}$ in (4.35). Using once more (5.46), (5.40), (5.38), (5.34), (5.25), we find for $\xi^+\xi$ the asymptotic form :

$$\xi^+\xi = \sum a_{nn'} r^{\varepsilon_n + \varepsilon_{n'} - 3} \quad (\text{pour } r \to \infty) \tag{5.54}$$

where, according to (5.46) the ε_n take the values $\varepsilon = \pm 1$, for the different components of Z, which leads to the following conclusions.

5. Physical conclusions from the behaviour, of a chiral state of a Dirac electron (a Majorana electron) , in an electric Coulombian field.

The asymptotic form (5.54) shows that $\xi^+\xi$ *would be integrable in the whole space only if* we had, in the sum of the second member of (5.54), not any number $\varepsilon_n = 1$. Because the different values of ε_n and $\varepsilon_{n'}$ give terms with : r^{-5} (for $\varepsilon_n + \varepsilon_{n'} = -2$); r^{-3} (for $\varepsilon_n + \varepsilon_{n'} = 0$); r^{-1} (for $\varepsilon_n + \varepsilon_{n'} = 2$).

Now, only the first type of terms gives a convergent integral at the infinite. In order that the integral of $\xi^+\xi$ converge, we must exclude the terms with $\varepsilon_n = 1$, which implies to annihilate the components Z_1 and Z_2 in the equation (5.42). But if we do so, we have identically $Z \equiv 0$ and the wave fonction disapears.

$\xi^+\xi$ is thus *never integrable* on the whole space. Therefore the Majorana electron has no bound states in a Coulomb field : the spectrum is continuous and there are only ionized states. It must be noticed that the sign of α in the equation (5.42) plays not any role : **the « Majorana electron » - more exactly the Majorana state of the Dirac electron – has a diffusive behaviour of the same type independently of a positive or negative charge of the coulomb field.**

It is easy to understand why this is so. In the state (5.25) of ξ which is associated with a value $\frac{l-1}{2}$ of the total cinetic moment, the terms corresponding to the different values m have, according to (5.33), exponential factors $e^{(-1)^m \omega t}$, where ω is the energy, so that ξ *is a superposition of states with positive and negative energies,* corresponding to the *"electron" or "positron" states.*

Thus we see that the Majorana theory is not a « simultaneous theory of the electron and of the positron ». It is only a hybrid-state of the Dirac electron, « which does not know » what is the sign of its electric charge. Thus we understand why it cannot be in a bound state. But its diffusing states will be quite different from the states of a fast "normal" electron state, because the wave functions are different from the wave functions of a keplerian system in a ionized state.

To make the fact better intelligible, we shall carry out the preceding calculation in the classical limit, and we shall see that all the trajectories are hyperbolic, as it might be guessed, but the hyperbolae are *not keplerian.* And given that the classical limit does not know the quantum superposition, there are two kinds of hyperbolae corresponding respectively to the diffusion, in an attractive or a repulsive field.

6. The geometrical optics approximation of the states of the Majorana electron.

Consider the general equation (5.14_1) for ξ, with the definitions (5.15) and the electromagnetic gauge (5.19). Now we introduce in (5.14_1) the following expression ($a(t,\vec{r})$ and $b(t,\vec{r})$ are new *spinors*) :

$$\xi = a(t,\vec{r})e^{-\frac{i}{\hbar}S(t,\vec{r})} + b(t,\vec{r})e^{+\frac{i}{\hbar}S(t,\vec{r})} \tag{5.55}$$

Neglecting the $\hbar$ terms, we have the equation :

$$\begin{aligned} &\left\{\left[\frac{1}{c}(\frac{\partial S}{\partial t} + eV) - (\nabla S - \frac{e}{c}\mathbf{A}).\mathbf{s}\right]a + m_0 c s_2 b^*\right\}e^{-\frac{i}{\hbar}S} \\ &-\left\{\left[\frac{1}{c}(\frac{\partial S}{\partial t} - eV) - (\nabla S + \frac{e}{c}\mathbf{A}).\mathbf{s}\right]b - m_0 c s_2 a^*\right\}e^{+\frac{i}{\hbar}S} = 0 \end{aligned} \tag{5.56}$$

For $\hbar \to 0$, the phases $\pm\frac{S}{\hbar}$ become infinitely fast and, multiplying alternatively the equation (5.56) by $e^{\frac{iS}{\hbar}}$ and $e^{\frac{-iS}{\hbar}}$, we find the geometrical optics approximation :

$$\begin{aligned} &\left[\frac{1}{c}(\frac{\partial S}{\partial t} + eV) - (\nabla S - \frac{e}{c}\mathbf{A}).\mathbf{s}\right]a + m_0 c s_2 b^* = 0 \\ &\left[\frac{1}{c}(\frac{\partial S}{\partial t} - eV) + (\nabla S + \frac{e}{c}\mathbf{A}).\mathbf{s}\right]b - m_0 c s_2 a^* = 0 \end{aligned} \tag{5.57}$$

Now we introduce a new spinor $\hat{b}$:

$$\hat{b} = s_2 b^* \tag{5.58}$$

Taking the complex conjugate second equation (5.57) multiplied on the left by s_2 (taking into account that s_2 is imaginery, which gives the plus sign in the second equation), one obtains for (5.57) :

$$\begin{aligned} &\left[\frac{1}{c}(\frac{\partial S}{\partial t} + eV) - (\nabla S - \frac{e}{c}\mathbf{A}).\mathbf{s}\right]a + m_0 c\hat{b} = 0 \\ &\left[\frac{1}{c}(\frac{\partial S}{\partial t} - eV) + (\nabla S + \frac{e}{c}\mathbf{A}).\mathbf{s}\right]\hat{b} + m_0 c\, a = 0 \end{aligned} \tag{5.59}$$

Multiplying the first equation by the matrix before $\hat{b}$ in the second one, we have :

$$\left\{\left[\frac{1}{c}(\frac{\partial S}{\partial t}-eV)+(\nabla S+\frac{e}{c}\mathbf{A}).\mathbf{s}\right]\left[\frac{1}{c}(\frac{\partial S}{\partial t}+eV)-(\nabla S-\frac{e}{c}\mathbf{A}).\mathbf{s}\right]-m_0^2c^2\right\}a=0 \quad (5.60)$$

or :

$$\left\{\begin{array}{l}\frac{1}{c}(\frac{\partial S}{\partial t}+eV)(\frac{\partial S}{\partial t}-eV)-(\nabla S+\frac{e}{c}\mathbf{A})(\nabla S-\frac{e}{c}\mathbf{A})-m_0^2c^2\\ +2\frac{e}{c}\left[V\nabla S+\frac{1}{c}\frac{\partial S}{\partial t}\mathbf{A}+i\nabla S\times\mathbf{A}\right].\mathbf{s}\end{array}\right\}a=0 \quad (5.61)$$

In order that : $a\neq 0$, we must set equal to zero the determinant of the matrix, which gives a Hamilton-Jacobi equation, that reads for $\vec{A}=0$:

$$\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\left(\nabla S\right)^2-\frac{e^2}{c^2}V^2-m_0^2c^2\right]^2-\frac{4e^2}{c^2}V^2\left(\nabla S\right)^2=0 \quad (5.62)$$

The factorization of the difference of two squares gives two e*quations* which take the following form in the coulomb case :

$$\frac{1}{c}\left(\frac{\partial S}{\partial t}\right)^2-\left(|\vec{\nabla}S|-\frac{\varepsilon e^2}{c}\frac{1}{r}\right)^2-m_0^2c^2=0 \quad (\varepsilon=\pm 1) \quad (5.63)$$

We can see that the sign of the charge does not play any role, because $\varepsilon=\pm 1$ is not due to the charge but to the factorization. And, still more important, these Hamilton-Jacobi equations are different from those which are found in the wellknown problem of an electron in a Coulomb field. In the latter case we have the following equations with two signs $\varepsilon=\pm 1$ too, but they are due to the sign of the charge and they correspond to two kinds of trajectories, ellipses or hyperbolae :

$$\frac{1}{c}^2\left(\frac{\partial S}{\partial t}-\frac{\varepsilon e^2}{r}\right)^2-(\vec{\nabla}S)^2-m_0^2c^2=0 \quad (\varepsilon=\pm 1) \quad (5.64)$$

Now, if we introduce in (5.63) the decomposition :

$$S=-Et+W \quad (5.65)$$

we find:

$$\frac{E^2}{c^2}-m_0^2c^2=\left[|\vec{\nabla}W|-\frac{\varepsilon e^2}{c}\frac{1}{r}\right]^2 \quad (5.66)$$

from which it follows immediately that :

$$E \geq m_0 c^2 \tag{5.67}$$

This means that **there are not any bound state and thus, no closed trajectories**. We have in (5.66) two equations :

$$(\nabla W)^2 = \frac{1}{c^2}\left(\sqrt{E^2 - m_0^2 c^4} + \frac{\varepsilon e^2}{r}\right)^2 \quad (\varepsilon = \pm 1) \tag{5.68}$$

and thus, in polar coordinates :

$$(\vec{\nabla} W)^2 = \left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \varphi}\right)^2 \tag{5.69}$$

Now, if we write :

$$W = J\varphi + f(r) \quad (J = \text{Const.}) \tag{5.70}$$

the equation (5.68) goes over into :

$$f(r) = \int \left(A^2 + \frac{2B}{r} + \frac{C}{r^2}\right)^{\frac{1}{2}} dr \tag{5.71}$$

with :

$$A = \frac{1}{c}\sqrt{E^2 - m_0^2 c^4}, \quad B = \frac{A\varepsilon e^2}{c}, \quad C = \frac{e^4}{c^2} - J^2 \tag{5.72}$$

The discriminant in (7.71) is positive :

$$\Delta' = B^2 - A^2 C = A^2 J^2 \geq 0 \tag{5.73}$$

So that the roots are real :

$$\frac{1}{r} = \frac{A(\frac{\varepsilon e^2}{c} \pm J)}{J^2 - \frac{e^4}{c^2}} \quad (\varepsilon = \pm 1) \tag{5.74}$$

We shall suppose now that $J \neq 0$ and given that we are in the limit of a quantum problem, we can write :

$$J \approx \hbar = \frac{e^2}{\alpha c} = 137\frac{e^2}{c} \gg \frac{e^2}{c} \tag{5.75}$$

This approximation is not essential, but it is convenient for what follows because we can write now the positive root (5.74) under the simplified form :

$$\frac{1}{r^*}=\frac{A}{J}\ \text{with} : r^*=\frac{Jc}{\sqrt{E^2-m_0^2c^4}} \tag{5.76}$$

So that the trajectory definite by (5.70) and (5.71) is now :

$$\frac{\partial W}{\partial J}=\varphi_0 \rightarrow \varphi-\varphi_0=J\int_{r^*}^{r}\frac{-1}{\left(A^2+\frac{2B}{r}+\frac{C}{r^2}\right)^{\frac{1}{2}}}\frac{dr}{r^2} \tag{5.77}$$

Taking $\varphi_0=0$, with the approximation (5.76) the equation of the trajectory becomes :

$$\frac{1}{r}=\frac{e^2\sqrt{E^2-m_0^2c^4}}{J^2c^2}\left(\varepsilon+\frac{Jc}{e^2}\cos\varphi\right)\quad(\varepsilon=\pm1) \tag{5.78}$$

It is a *hyperbola*, because in virtue of (5.75), its eccentricity is greater than one :

$$\frac{Jc}{e^2}>1 \tag{5.79}$$

It must be underlined, that the hyperbolic character of the trajectory was already determinate by the (5.67) and not only by the simplified form (5.75). In conclusion, there are not any bound state as it was noticed in advance, but we should not forget that there are two possible types of trajectories, because $\varepsilon=\pm1$, the two signs corresponding to the two equations (5.63) :

- If $\varepsilon=+1$, the *concavity* of the trajectory is oriented to the central field and the motion is *attractive*.

- If $\varepsilon=-1$, the *convexity* of the trajectory is oriented to the central field and the motion is *repulsive*.

Therefore, in accordance with the quantum treatement, both cases are possible, whatever could be the charges and the central-field.

It is interesting to compare our results with the classical case of a relativistic electron in a coulombian potential : we consider the classical equation (5.64) again, introducing (5.69) and (5.70), which gives an integral of the same form as (5.71) :

$$f(r)=\int\left(A^2+\frac{2B'}{r}+\frac{C}{r^2}\right)^{\frac{1}{2}}dr \tag{5.80}$$

$$A=\frac{1}{c}\sqrt{E^2-m_0^2c^4}\ ,\ B'=\frac{E\varepsilon e^2}{c^2}\ ,\ C=\frac{e^4}{c^2}-J^2 \tag{5.81}$$

In the *case* : $(E \geq m_0c^2)$, in comparison with (5.72), one can see, that the only coefficient B remains, while the factor A is substituted by $\frac{E}{c}$, which means the coincidence of these two cases for the limit $E \to m_0c^2$. But it must be noticed that, in the preceding case, the condition (5.67) : $E \geq m_0c^2$ was necessary, while here, in the classical case, it is only one of two possibilities because we could have $E < m_0c^2$, which would correspond to elliptic trajectories (i.e. bound states).

Taking the preceding calculation again with the constants (5.81), we find the trajectories :

$$\frac{1}{r} = \frac{e^2E}{J^2c^2}\left(\varepsilon + c\frac{\sqrt{(E^2 - m_0^2c^4)J^2 + m_0^2c^2e^4}}{Ee^2}\cos\varphi\right) \tag{5.82}$$

This formula, which is good only for : $E > m_0c^2$, differs from the classical formula only by the absence of the precession factor in the argument of the cosine that we have neglected in virtue of (5.75) and the preceding approximation (which actually results in the sustitution C by $-J^2$). On the contrary the approximation would not be valuable under the root-sign in the expression of the excentricity, excepted if $E >> m_0c^2$, which is the limit to which tend the expressions (5.78) and (5.82).

But the interesting case arises when $E - m_0c^2$ is small, because the eccentricity of the classical hyperbola depends on E and :

$$c\frac{\sqrt{(E^2 - m_0^2c^4)J^2 + m_0^2c^4}}{Ee^2} \to 1 \quad \text{if} \quad E \to m_0c^2 \tag{5.83}$$

Thus, the classical *parabolic* trajectory results when : $E \to m_0c^2$.

On the contrary, the exccentricity of the hyperbola (7.78), is independent from energy and consequently from the angle between the asymptotes, while :

$$\frac{1}{p} = \frac{e^2\sqrt{E^2 - m_0^2c^4}}{J^2c^2} \to 0 \quad \text{if} \quad E \to m_0c^2 \tag{5.83'}$$

Therefore, the parameter approches infinity, while (5.82) shows that in the classical case, when $E \to m_0c^2$, the parameter tends to a finite value. Consequently, for low energies, we find two different ways of behaviour which could be experimentally distinguished provided that one could create this strange constrainted state of the electron described by the Majorana field.

7. How could one observe a Majorana electron ?

We have seen that in a coulomb field, at the geometrical optics approximation, , the Majorana electron behaves either like a particle with a negative charge, or like a particle with a positive charge, but it remains different from an electron or a positron because its motion is not keplerian.

Nevertheless this is only the problem of trajectories, that is to say the problem of the rays of the wave, given by the Jacobi equation. If we introduce the corresponding approximate expression of the action S in the expression (5.55) of the wave function we find an approximate solution of the equations (5.57).

So, we shall find that, despite that trajectories seem "choose" their charge (+ or -), the wave function evidently remains a superposition state of two waves with *opposite phases,* i.e. waves with conjugated charges. Let us apply that to plane waves.

We write (5.55) with constant spinors a and b :

$$\xi = a\, e^{i(\omega t - \vec{k}.\vec{r})} + b\, e^{-i(\omega t - \vec{k}.\vec{r})} \tag{5.84}$$

and we introduce the above expression in (5.14₁) with V = A = 0, and an angle θ which is definite in (5.19). Analogously to the one of the § 6 a simple computation gives :

$$\frac{\omega^2}{c^2} = k^2 + \frac{m_0^2 c^2}{\hbar^2} \tag{5.85}$$

$$\xi = a\, e^{i(\omega t - \vec{k}.\vec{r})} - \frac{\hbar}{m_0 c}\left(\frac{\omega}{c} - \vec{s}.\vec{k}\right) s_2 a^*\, e^{-i(\omega t - \vec{k}.\vec{r})} \tag{5.86}$$

This is a superposition of two waves with energies of opposite signs. But let us go back to the Dirac equation, i.e. to the couple of two equations (5.14) linked by (5.6), with the condition (5.19). Therefore, it is not exactly the Majorana field but the Dirac field constrainted by (5.5). In other words it is the equation (5.12) with the value (5.19) for the angle θ, and $A_\mu = 0$.

Now we must find the wave ψ owing to (5.86) and :

$$\eta = s_2 \xi^*, \quad \psi = \frac{1}{\sqrt{2}} \begin{bmatrix} \xi + \eta \\ \xi - \eta \end{bmatrix} \tag{5.87}$$

We shall take Oz for the direction of propagation of the wave and :

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \vec{k} = \{0, 0, k\} \tag{5.88}$$

a_1 and a_2 = components of a, in (5.86). So we find :

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \tag{5.89}$$

$$\psi_1 = a_1 \begin{bmatrix} 1+\frac{\hbar}{m_0c}(\frac{\omega}{c}+k) \\ 0 \\ 1-\frac{\hbar}{m_0c}(\frac{\omega}{c}+k) \\ 0 \end{bmatrix} e^{i(\omega t - kz)} - ia_1^* \begin{bmatrix} 0 \\ 1+\frac{\hbar}{m_0c}(\frac{\omega}{c}+k) \\ 0 \\ -[1-\frac{\hbar}{m_0c}(\frac{\omega}{c}+k)] \end{bmatrix} e^{-i(\omega t - kz)} \tag{5.90}$$

$$\psi_2 = a_2 \begin{bmatrix} 0 \\ 1+\frac{\hbar}{m_0c}(\frac{\omega}{c}-k) \\ 0 \\ 1-\frac{\hbar}{m_0c}(\frac{\omega}{c}-k) \end{bmatrix} e^{i(\omega t - kz)} + ia_2^* \begin{bmatrix} 1+\frac{\hbar}{m_0c}(\frac{\omega}{c}-k) \\ 0 \\ -[1-\frac{\hbar}{m_0c}(\frac{\omega}{c}-k)] \\ 0 \end{bmatrix} e^{-i(\omega t - kz)} \tag{5.91}$$

ψ is the superposition of two waves ψ_1 and ψ_2 with the constants a_1 and a_2 . Each wave ψ_1 and ψ_2 depends on energy and helicity, which is easy to define because, if Oz is the direction of propagation, so that the spin is projected in the same direction and :

$$\sigma_3 = \begin{bmatrix} s_3 & 0 \\ 0 & s_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{5.92}$$

We see that :

1. ψ_1 is a superposition of two waves with the same sign of helicity and charge (respectively + and – for each wave).

2. ψ_2 is a superposition of two waves with opposite helicities and charges.

The relative phase of the components of ψ_1 or ψ_2 (i.e. of $a_{1,2}$ and $a_{1,2}^*$) has no physical meaning, because the constant θ in (5.12) or (5.14) is arbitrary. Now, for low energies :

$$|k| \ll \frac{\omega}{c}, \quad \frac{\omega}{c} = \frac{m_0c}{\hbar} \tag{5.93}$$

we have in a first approximation :

$$\psi_1 = \begin{bmatrix} a_1 e^{i(\omega t - kz)} \\ -i a_1^* e^{-i(\omega t - kz)} \\ 0 \\ 0 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} i\, a_2 e^{-i(\omega t - kz)} \\ a_2^* e^{i(\omega t - kz)} \\ 0 \\ 0 \end{bmatrix} \qquad (5.94)$$

In conclusion, if we could « keep alive», i.e. without annihilation, during a sufficiently long time two parallel beams of electrons and positrons with the same energy and the above polarisation, the couples so definite would have the behaviour of a Majorana electron. In particular in a Coulomb field an electron in such a state would have the strange behaviour described in the preceding paragraph, instead of the classical Kepler laws.

8. The equation in the magnetic case.

We have recalled in (5.3) the general nonlinear equation of a magnetic monopole, and we know that the chiral gauge invariance is broken and the magnetic charge is no more conserved if we add a linear mass term (it is the reason for which the Dirac equation does not conserve the magnetic charge). As we know (§ 3), the Majorana condition insures the conservation of chiral currents and thus, of magnetism.

Such an equation is not really chiral gauge invariant but in the present case, it admits a *subset of gauge invariant solutions*. Now, remember that the chiral invariance is an invariance, with respect to the rotations in the chiral plane $\{\omega_1, \omega_2\}$, i.e. with respect to the rotations of an angle A, which can be obtained in two ways :

a) The first way is to introduce in the Lagrangian a mass term which only depends on the norm of the vector $\{\omega_1, \omega_2\}$; this was done until now, and this gives the equation (5.3).

b) The second way is to add to the Lagrangian of the linear monopole an arbitrary mass term, which is not necessarily chiral invariant (as was the norm of $\{\omega_1, \omega_2\}$), but which is so, that the obtained equation has a subset of solutions which annihilates the chiral invariant :

$$\rho = (\omega_1^2 + \omega_2^2)^{1/2} = 0 \qquad (5.95)$$

Such solutions thus obey the generalized Majorana condition (5.6), that we write here in a more simple form :

$$\Psi = e^{i\theta}\gamma_2\psi^* = e^{i\theta}\psi_c \qquad (5.96)$$

Actually, we can put $\theta = 0$, which will be done a little bit later. A priori, we could start from an arbitrary term of mass, but we shall choose for the sake of simplicity the linear mass term of the Dirac equation. So that we shall introduce in the equation of the massless monopole, the mass term of the equation (2.1) under the condition (5.95) or (5.96), which will be expressed by means of a Lagrange multiplier. Thus, we have the lagrangian :

$$L = \overline{\Psi}\gamma_\mu[\partial_\mu]\Psi - g\hbar c\overline{\Psi}\gamma_\mu\gamma_5 G_\mu\Psi - m_0 c\hbar\overline{\psi}\psi + \lambda(\omega_1^2 + \omega_2^2) \qquad (5.97)$$

from which, varying $\bar{\psi}$, we deduce the following equation which looks like our nonlinear equation in (Ch. 4), with a linear term :

$$\gamma_\mu(\partial_\mu - g\hbar c\gamma_5 G_\mu)\Psi - m_0 c\hbar\psi + \lambda(\omega_1 - i\omega_2\gamma_5)\Psi = 0 \tag{5.98}$$

The difference between this equation and the equation of our nonlinear monopole is the presence of a linear mass term and of the constant λ instead of $m(\rho^2)$. But the linear term will be transformed, and the nonlinear term itself will disappear because we must vary L with respect to the Lagrange multiplier λ , in order to find (5.95). Thus we have :

$$\omega_1 = \omega_2 = 0 \tag{5.99}$$

This gives the relation (5.97) and annihilates the λ term in (5.98). The Lagrange multiplier thus remains undetermined, since it does not appear in the equation. If we introduce (5.96), **we find the Majorana equation up to a phase factor** $e^{i\theta}$, with a magnetic interaction instead of an electric one :

$$\gamma_\mu(\partial_\mu - g\hbar c\gamma_5 G_\mu)\Psi - m_0 c\hbar e^{i\theta}\gamma_2\psi^* = 0 \tag{5.100}$$

It is a new nonlinear equation of a magnetic monopole, different from the one found in the Chapter 5. In the Weyl représentation (Ch. 3), the equation (5.100) splits into two equations, formally separated, but actually linked among each other :

$$\begin{aligned}(\pi_0^+ + \pi^+ \cdot s)\xi - im_0 c e^{i\theta} s_2 \xi^* = 0 \\ (\pi_0^- - \pi^- \cdot s)\eta + im_0 c e^{i\theta} s_2 \eta^* = 0\end{aligned} \tag{5.101}$$

with the definitions :

$$\begin{aligned}\pi_0^+ = \frac{1}{c}(i\hbar\frac{\partial}{\partial t} + gW), \quad \pi^+ = -i\hbar\frac{\partial}{\partial t} + \frac{g}{c}\mathbf{G} \\ \pi_0^- = \frac{1}{c}(i\hbar\frac{\partial}{\partial t} - gW), \quad \pi^- = -i\hbar\frac{\partial}{\partial t} - \frac{g}{c}\mathbf{G}\end{aligned} \tag{5.102}$$

We can remark that, *in the electric case, we had only one operator* : $\{\pi_0, \pi\}$, while *in the magnetic case, we have two operators : right and left*. Before to examine the equations (5.101), we stop, to specify some points concerning the equation (5.98).

First of all, this equation was found a long time ago by Hermann (*Weyl*), only *for a free wave, i.e. without interaction,* and with another aim. For Weyl, the nonlinear term was not a Lagrange condition. It was a change of the Dirac equation, owing to which the nonlinear Weyl equation (contrary to the Dirac linear equation) has the property of keeping the same form, in general relativity, would it be expressed, in the metric form with an affine connection $\Gamma_{\mu\lambda\nu}$ depending on the $g_{\mu\nu}$, or with coefficients $\Gamma_{\mu\lambda\nu}$ independent of the $g_{\mu\nu}$.

In various forms the same equation without interactions was later found again by several authors and reexamined from different points of view. Two papers are particularly interesting with respect to our problem :

a) We have already seen the first of them (*Rodichev*) in the Ch. 4 on torsion. Let us just recall that it is based on a particular case of (5.98), where λ is an ordinary constant :

$$\gamma_\mu \partial_\mu \Psi + \lambda(\Omega_1 - i\Omega_2\gamma_5)\Psi = 0 \tag{5.103}$$

It was shown (*Lochak 5* and *Ch. 4*) that the chiral invariant is equal, up to a constant factor, to the total curvature. But in our case the space is flat and the curvature is reduced to the torsion, so that when we show that the Majorana condition (5.96) is equivalent to the condition (5.95) it signifies actually that the Majorana condition, annihilates the torsion of the space.

b) Now we give results due to A. Bachelot (*Bachelot 1,2, 3*), wo solved the global Cauchy problem for the equation (5.103) without electromagnetic interaction, but with initial conditions which are not supposed to be small : they are only such that the chiral invariant $\rho = (\omega_1^2 + \omega_2^2)^{1/2}$ is small. In other words, it remains in the vicinity of the condition (5.95) which implies, as we know, to be close to the generalized Majorana condition.

To prove his theorem, Bachelot proved at first the following lemma, of great interest in itself :

- Consider the Dirac equation without interaction, but with a mass term M, possibly depending on space and time :

$$\gamma_\mu \partial_\mu \Psi + M\psi = 0 \tag{5.104}$$

Bachelot proved that if the chiral invariant $\rho = (\omega_1^2 + \omega_2^2)^{1/2}$ vanishes, at a given instant, in the whole space, it remains equal to zero later. It is easy to generalize the lemma of Bachelot in the presence of a *magnetic interaction* and we shall directly formulate and prove it in this more general case :

- Given the equation :

$$\gamma_\mu(\partial_\mu \Psi - g\hbar c\gamma_5 G_\mu)\Psi - m_0 c\hbar\psi = 0 \tag{5.105}$$

if at a given instant the chiral invariant $\rho = (\Omega_1^2 + \Omega_2^2)^{1/2}$ (and so, the torsion of the space) vanishes in the whole space, il remains equal to zero. Bachelot starts from two conservation laws :

$$\partial_\mu \overline{\psi}\gamma_\mu\psi = 0,\ \partial_\mu \overline{\tilde{\psi}}\gamma_2\gamma_4\gamma_\mu\psi = 0 \quad (\tilde{\psi} = \text{transposed}\ \psi) \tag{5.106}$$

The first law is the conservation of the Dirac current, i.e of electricity. It must be noticed that the chiral currents are not separately conserved, contrary to (3.7), because of the presence of a linear mass term in (5.105) ; but their sum is conserved like in the Dirac equation and this sum is precisely the Dirac electric current that appers in (5.106).

The second law is the conservation of the crossed current between charge conjugated states. Bachelot deduced it from the equation (5.104), but it is true also for the equation (5.105) **with a magnetic**

interaction. On the contrary the second conservative law would be wrong in the case of an ordinary Dirac equation with an electric interaction. We get indeed in this case :

$$\partial_\mu \tilde{\psi}\gamma_2\gamma_4\gamma_\mu\psi + iA_\mu\tilde{\psi}\gamma_2\gamma_4\gamma_\mu\psi = 0 \tag{5.107}$$

Now, if these two laws (5.106) are true, Bachelot uses the conservation laws which entail :

$$\int_{\mathbf{R}^3} |\psi|^2\, dx = \text{Const}, \quad \int_{\mathbf{R}^3} \tilde{\psi}\gamma_2\psi\, dx = \text{Const} \tag{5.108}$$

provided that these integrals do exist. This reservation must be demanded because we know that there are no bound states between an electric and a magnetic charge (*Lochak 3, 4*), so that this result is not general.

Under the preceding restriction, we find from (5.108) :

$$\int_{\mathbf{R}^3} |\Psi - e^{i\theta}\gamma_2\psi^*|^2\, dx = 2\int_{\mathbf{R}^3} \{|\Psi|^2 - \Re e^{-i\theta}\tilde{\psi}\gamma_2\psi^*\} dx = \text{Const} \tag{5.109}$$

Therefore, **if at a given instant the condition (5.95), or equivalently (5.96), is realized it will be realized in the future**. This is the lemma of Bachelot, and we know that it is true not only for the equation (5.104) but also for the equation (5.105).

If the preceding formulae are true, the condition to which the equations (5.100), (5.101) were submitted through the Lagrange multipliers, will be strongly weakened, because instead of a constraint imposed at every instant, we have only an initial condition. Therefore, **the Majorana magnetic states are simply particular solutions of the Dirac equation with a magnetic interaction**.

More precisely, these states are *monopole states of the Dirac equation*, because it will be shown that the equations (5.100) or (5.101) are actually chiral invariant, despite that (5.105) is not chiral invariant. They represent a couple of monopoles and, in order to make them to appear, it is sufficient, at least in certain cases, to satisfy an initial condition.

Let us underline, once more, that in virtue of (5.107) this conclusion, which is true in the magnetic case, is not in the electric case. So that, **if we are able to fatisfy the conditions (5.95), we shall obtain monopôles but not electrons.**

10. Another possible equation. The gauge invariance problem.

Let us introduce the transformation : $\Psi \to e^{ig\hbar c\gamma_5\Phi}\Psi$ (see Ch. 4) in the equation (5.100).We find :

$$\gamma_\mu[\partial_\mu\Psi - g\hbar c\gamma_5(G_\mu + i\partial_\mu\Phi)]e^{ig\hbar c\gamma_5\Phi}\Psi - m_0c/\hbar e^{i\theta}\gamma_2 e^{-ig\hbar c\gamma_5\Phi}\psi^* = 0 \tag{5.110}$$

And then, taking into account the anticommutation rules of γ matrices :

$$\gamma_\mu\left[(\partial_\mu\Psi - g\hbar c\gamma_5(G_\mu + i\partial_\mu\Phi)\right]\Psi - m_0c/\hbar e^{i\theta}\gamma_2\psi^* = 0 \tag{5.111}$$

We find the correct interaction term with the G_μ potentials, but with a phase factor Φ, the origin of this factor is the angle A. The chiral gauge invariance is not obvious as it could be expected, because this invariance in its general form appears only in the equations in which the chiral angle A does not appear, while in the present case, we started from the equation (5.105), which is not chiral gauge invariant. We have just imposed one of the condition (5.95), which does not make disappear the angle A, but undetermined : it appears in the equation, but its value can be eliminated it is a polar angle around a nil rotation vector.

Finally, the preceding phase factor is eliminated by a choice of the angle A because, any way, the gauge invariance is lost, and the phase θ may be eliminated too because it plays no dynamical rôle. And thus we can write, as a consequence of (5.95) :

$$\Psi = \gamma_2\psi^* = \psi_c\xi = is_2\eta^*\eta = -is_2\xi^* \qquad (5.112)$$

And instead of (5.100) and (5.101) we have respectively (without θ) :

$$\gamma_\mu(\partial_\mu - g\hbar c\gamma_5 G_\mu)\Psi - m_0 c\hbar\gamma_2\psi^* = 0 \qquad (5.113)$$

and [see (5.102)] :

$$\begin{aligned}(\pi_0^+ + \pi^+ \cdot s)\xi - im_0cs_2\xi^* &= 0\\ (\pi_0^- + \pi^- \cdot s)\eta + im_0cs_2\eta^* &= 0\end{aligned} \qquad (5.114)$$

11. Geometrical optics approximation.

Until now, all seems well, but it would be desirable to test the qualities of the preceding equations for a well known case, as for instance the interaction of a monopole with an electric charge, as it has been done above in the comparable case of the electron, replacing (5.14) by (5.114), it seems quite easy because of the apparent identity of both equations. Unfortunately, **it is illusory** because the potentials hidden in these formulae are fundamentally different, so that the magnetic case is far more complicated than the electric one. And the present case is more difficult than the case of a linear massless monopole (Ch.3), precisely because of the nonlinear mass term.

For these reasons, we shall content ouselves with the classical approximation. Thus we shall take the equations (5.114) with π^+ and π^- definite in (5.102), with the following expressions :

$$\xi = a\exp(-iS/\hbar) + b\exp(iS/\hbar), \quad \eta = -is_2\xi^* \qquad (5.115)$$

A calculation analogous to the one in § 6 gives an equation of the "Hamilton-Jacobi" type :

$$[(\frac{1}{c}\frac{\partial S}{\partial t})^2 - (\nabla S + \frac{g\mathbf{G}}{c})^2 - m_0^2c^2][(\frac{1}{c}\frac{\partial S}{\partial t})^2 - (\nabla S - \frac{g\mathbf{G}}{c})^2 - m_0^2c^2] = 4m_0^2g^2\mathbf{G}^2 \qquad (5.116)$$

It is quite different from the equation (5.61) because of the difference of potentials (see : *(1.26)*) :

$$W = 0, \quad G_x = \frac{e}{r}\frac{yz}{x^2+y^2}, \quad G_y = \frac{e}{r}\frac{-xz}{x^2+y^2}, \quad G_z = 0, \quad r = \sqrt{x^2+y^2+z^2} \qquad (5.117)$$

The electric Coulomb-field is written as :

$$\mathbf{E} = rot\,\mathbf{G} = e\mathbf{r} / r^3 \tag{5.118}$$

Now, it must be recognized that (5.116) is not the equation of a classical magnetic monopole in the presence of an electric charge which would be :

$$\mathbf{E} = rot\,\mathbf{G} = e\mathbf{r} / r^3$$

$$either : [(\partial S / \partial t)^2 / c^2 - (\nabla S + g\mathbf{G} / c)^2 - m_0^2 c^2] = 0 \tag{5.119}$$

$$or : [(\partial S / \partial t)^2 / c^2 - (\nabla S - g\mathbf{G} / c)^2 - m_0^2 c^2] = 0 \tag{5.120}$$

According to the sign of the magnetic charge. The simultanuous presence of both brackets $(\nabla S + g\mathbf{G} / c)$ and $(\nabla S - g\mathbf{G} / c)$, in (5.116), suggests that the equation contains a couple of monopoles of opposite signs, and one can observe that, far from the center of the electric charge : $G \to 0$ and that the equation (5.116) splits into two composantes (5.119) and (5.120). Thus, actually, we find, **asymptotically**, a couple of classical monopoles. This may be called the zero approximation. The first order equations (nearer to the center) may be written as :

$$\begin{aligned} &[(\partial S / \partial t)^2 / c^2 - (\vec{\nabla} S + g\mathbf{G} / c)^2 - m_0^2 c^2] = 2m_0 g\,|\,\mathbf{G}\,| \\ &[(\partial S / \partial t)^2 / c^2 - (\vec{\nabla} S - g\mathbf{G} / c)^2 - m_0^2 c^2] = -2m_0 g\,|\,\mathbf{G}\,| \end{aligned} \tag{5.121}$$

these equations have additive terms with respect to (5.119) and (5.120).

It is interesting to introduce, in one of these equations, the two following quantities :

$$p = \nabla S = m\frac{d\mathbf{r}}{dt}, \quad \lambda = \frac{egc}{\varepsilon} \quad (\varepsilon = \text{energy}) \tag{5.122}$$

which gives the following equation :

$$d^2 r / dt^2 = -\lambda / r^3 \cdot dr / dt \times r - m_0 g\,|\,\mathbf{G}\,| \tag{5.123}$$

Without the second term of the second member, it would be the Poincaré equation (1.2), of the interaction between an electric and a magnetic charge in classical mechanics. Remember that we have already obtained such an equation, at the geometrical optics limit of our equation of a massless monopole (Ch. 3).

Nevertheless, we cannot neglect this strange additional term that appears in (5.123), and which is the same term as in (5.116), (5.121) and (5.122) : a kind of remembrance of the « sin » of the introduction of a linear massterm, which disappears far from the center of charge, but which calls for a certain cautiousness concerning the "Majorana monopole".

It must be added that the importance of the Poincaré equation is not only due to the fame of its author, but to the fact that this equation is experimentally verified by the Birkeland effect (Ch.1) ! It is the equation of the motion of a beam of cathodic rays in the presence of a pole of a linear magnet : actuallly a magnetic monopole. This is why we have attached a great importance to the fact that the Poincaré equation is the classical limit of our equation of the magnetic monopole.

It is thus impossible to be indifferent to a violation of the Poincaré equation. Yet it is not a total invalidation of the Majorana monopole because the additive term is always the same one and it tends to zero in two cases :

a) If the proper mass tends to zero, therefore *if this monopole tends to our massless monopole*. But this case is not interesting because *it is evident by* (5.113) *and* (5.114) *that the nonlinear Majorana term tends to zero.*
b) *Far from the center of charge*, which is due to the potentials and which is not evident from (5.113) and (5.114). This gives to the Majorana monopole an asymptotic significance. However it must be noticed that, far from the center, the additive terms become negligible, but unfortunately, the potential terms become negligible too, so that *the Majorana monopole tends to our massless monopole when it is no more a monopole !*

APPENDIX A

Let us give a proof of the formulae (4.6). First of all, we know that, by the very definition of $\Omega_l^m(-)$ *and* $\Omega_l^m(+)$, we have :

$$\vec{J}^2\Omega_{l-1}^m(+) = j(j+1)\Omega_{l-1}^m(+), \quad j = l - \frac{1}{2} \tag{A1}$$

$$\vec{J}_z\Omega_l^m(-) = u\Omega_l^m(-), \quad \vec{J}_z\Omega_{l-1}^m(+) = u\Omega_{l-1}^m(+), \quad u = m - \frac{1}{2} \tag{A.2}$$

where we have :

$$\mathbf{J} = \mathbf{L} + \mathbf{s}, \quad \mathbf{L} = -i\,\mathbf{r} \times \nabla \tag{A.3}$$

Now, we can easily verify that the operator :

$$\vec{\mathbf{s}}.\vec{\mathbf{n}} = \frac{1}{r}\vec{\mathbf{s}}.\vec{\mathbf{r}} = \begin{bmatrix} \cos\theta & sin\theta e^{-i\varphi} \\ sin\theta e^{i\varphi} & -\cos\theta \end{bmatrix} \tag{A.4}$$

commutes with $\mathbf{J}$:

$$[\vec{\mathbf{J}}, \vec{\mathbf{s}}.\vec{\mathbf{n}}] = 0 \tag{A.5}$$

Therefore, $\vec{\mathbf{s}}.\vec{\mathbf{n}}$ transforms a subspace Ω that belongs to the subspace of eigenvalues of J^2 and J_z in an element of the same subspace. We have, for instance :

$$\vec{\mathbf{s}}.\vec{\mathbf{n}}\,\Omega_l^m(+) = A\Omega_l^m(+) + B\Omega_{l+1}^m(-) \tag{A.6}$$

where the constants A and B do not depend on *m*. We shall compute them for particular values of *m and of the polar angles :*

$$m = l + 1, \quad \theta = \frac{\pi}{2}, \quad \varphi = 0 \tag{A.7}$$

From (5.21), we have :

$$\Omega_l^{l+1}(+) = \begin{bmatrix} Y_l^l(\frac{\pi}{2},0) \\ 0 \end{bmatrix}, \quad \Omega_{l+1}^{l+1}(-) = \begin{bmatrix} \left(\frac{l}{2l+3}\right)^{12} Y_{l+1}^l(\frac{\pi}{2},0) \\ -\left(\frac{2l+2}{2l+3}\right)^{12} Y_{l+1}^{l+1}(\frac{\pi}{2},0) \end{bmatrix} \tag{A.8}$$

Now, from (5.22) :

$$Y_{l+1}^l(\frac{\pi}{2},0), \quad \left(\frac{2l+2}{2l+3}\right)^{12} Y_{l+1}^{l+1}(\frac{\pi}{2},0) = -Y_l^l(\frac{\pi}{2},0) \tag{A.9}$$

and (A.4) gives :

$$\vec{s}.\vec{n}\,(\frac{\pi}{2},0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{A.10}$$

Finally, it is sufficient to introduce (A.8), (A.9) and (A.10) in (A.7), to find :

$$A = 0, \quad B = 1 \tag{A.11}$$

which proves the first relation (5.23). The second relation is evident because :

$$(\vec{\mathbf{s}}.\vec{\mathbf{n}})^2 = I \tag{A.12}$$

we have :

$$Y_l^{m*}(\theta,\varphi) = (-1)^m Y_l^{-m} \tag{A.13}$$

APPENDIX B

To prove (5.29), remember that in (A.3), $\vec{L}$ and $\vec{s}$ commute, so from (5.21) and (A.1), we have :

$$\vec{J}^2\Omega_l^m(\pm) = j(j+1)\Omega_l^m(\pm), \quad \vec{L}^2\Omega_l^m(\pm) = l(l+1)\Omega_l^m(\pm)$$
$$\vec{S}^2\Omega_l^m(\pm) = s(s+1)\Omega_l^m(\pm) = \frac{3}{4}\Omega_l^m(\pm). \tag{B.1}$$

Thus, applying (A.3), we have :

$$(\mathbf{L}+\mathbf{S})^2\Omega_l^m(\pm) = (\mathbf{L}^2 + \mathbf{S}^2 + 2\vec{\mathbf{L}}.\vec{\mathbf{S}})\Omega_l^m(\pm) = (\mathbf{L}^2 + \mathbf{S}^2 + \vec{\mathbf{L}}.\vec{\mathbf{S}})\Omega_l^m(\pm) \tag{B.2}$$

so that (B.1) gives : $j(j+1)\,\Omega_l^m(\pm) = [l(l+1) + \frac{3}{4} + \vec{\mathbf{L}}.\vec{\mathbf{s}}]\ \Omega_l^m(\pm)$ and the relations (5.29).