PART ONE

Theory of the Leptonic Monopole

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Contents

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CHAPTER 1

Theoretical Background

1. THEORIES OF POINCARÉ, DIRAC, AND CURIE

1.1 The Birkeland-Poincaré effect

In 1896, Kristian Birkeland introduced a straight magnet in a Crookes tube, and he was puzzled by a convergence of the cathodic beam that did not depend on the orientation of the magnet (Birkeland, 1896). Henri Poincaré explained this effect by the action of a magnetic pole on the electric charges of the beam (such charges were only conjectured at that time); he showed that it may be due to the action of only one pole of the magnet and that, for symmetry reasons, it must be independent of the sign of the pole (Poincaré, 1896). (See Figure 1.1)

To describe this effect, Poincaré wrote down the equation of an electric charge in a coulombian magnetic field created by one end of the magnet. The magnetic field is expressed as

$$\mathbf{H} = g\frac{1}{r^2}\mathbf{r}, \tag{1.1}$$

where g is the magnetic charge. From the expression of the Lorentz force (Poincaré, 1896) the following equation results:

$$\frac{d^2\mathbf{r}}{dt^2} = \lambda\frac{1}{r^3}\frac{d\mathbf{r}}{dt}\times\mathbf{r}; \quad \lambda = \frac{eg}{mc}, \tag{1.2}$$

where e and m are the electric charge and the mass of the electron.

Poincaré found the following integrals of motion, where A, B, C, and Λ are arbirary constants:

$$\mathbf{r}^2 = Ct^2 + 2Bt + A; \quad \left(\frac{d\mathbf{r}}{dt}\right)^2 = C. \tag{1.3}$$

$$\mathbf{r}\times\frac{d\mathbf{r}}{dt} + \lambda\frac{\mathbf{r}}{r} = \mathbf{\Lambda} \tag{1.4}$$

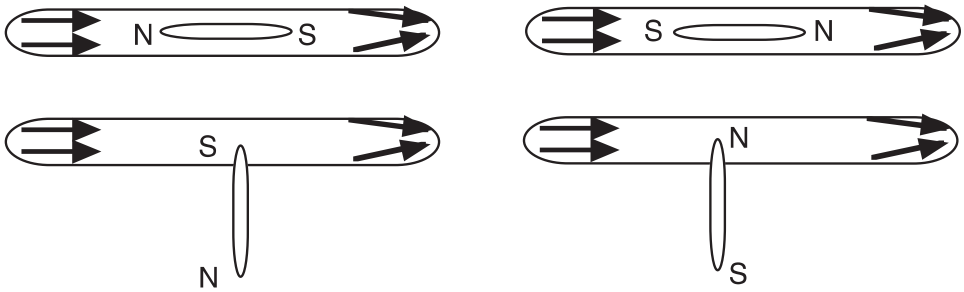

Figure 1.1 The Birkeland-Poincaré effect. When a straight magnet is introduced in a Crookes tube, the cathodic rays converge regardless of the orientation of the magnet. Above: the cases considered by Birkeland. Below: the cases corresponding to the calculations of Poincaré.

He obtained the following from Eqs. (1.4) and (1.2):

$$\boldsymbol{\Lambda}.\mathbf{r} = \lambda r; \quad \frac{d^2\mathbf{r}}{dt^2}\cdot\mathbf{r} = \frac{d^2\mathbf{r}}{dt^2}\cdot\frac{d\mathbf{r}}{dt} = 0. \tag{1.5}$$

This means that **r** describes an axially symmetric cone—the *Poincaré cone*, 1896—and that the acceleration is perpendicular to its surface, so that **r** follows a *geodesic line*. If the cathodic rays are emitted far from the magnetic pole, with a velocity V parallel to the z-axis, they will have an asymptote that obeys the following equations:

$$x = x_0; \quad y = y_0 \tag{1.6}$$

And we find, from Eqs. (1.3) and (1.4):

$$C = V^2; \quad \boldsymbol{\Lambda} = \{y_0 V, \ -x_0 V, \ \lambda\}. \tag{1.7}$$

Thus the z-axis is a generating line of the Poincaré cone, the half-vertex angle $\boldsymbol{\Theta}'$ of which is given by

$$\sin \boldsymbol{\Theta}' = \frac{V}{\lambda}\sqrt{x_0^2 + y_0^2} \tag{1.8}$$

Now, after the emission, the cathodic ray becomes a geodesic line rotating along the cone and crosses the z-axis at distances from the origin given by

$$\frac{\sqrt{x_0^2 + y_0^2}}{\sin \phi}; \quad \frac{\sqrt{x_0^2 + y_0^2}}{\sin 2\,\phi}; \quad \frac{\sqrt{x_0^2 + y_0^2}}{\sin 3\,\phi}; \ldots \quad \phi = 2\pi \sin \boldsymbol{\Theta}' \tag{1.9}$$

Therefore, if the emitting cathode is a small disk of radius $\sqrt{x_0^2 + y_0^2}$, orthogonal to the z-axis and if the position of the magnetic pole is such that one of these points is on the surface of the cone, there will be a concentration of electrons emitted by the periphery of the cathode and even, approximately, by the whole disk: this is the *focusing* effect observed by Birkeland.

This is an important result because the Poincaré equation [Eq. (1.2)] and the integral of motion [Eq. (1.4)] may be seen as experimentally verified because, for electrons falling on a fixed monopole, it is proved by the Birkeland effect. Conversely, for monopoles falling on a fixed-coulomb electric charge, it is implicitly proved by the simple fact that the interacting force is the same for electricity and magnetism. Consequently, the Poincaré equation remains true.

In Eq. (1.4), the first term is clearly the orbital momentum of the electron with respect to the magnetic pole. The second term was later interpreted by J. J. Thomson (Thomson, 1904 and Lochak, 1995b), who showed that

$$\frac{eg}{c}\frac{\mathbf{r}}{r} = \frac{1}{4\pi c}\int_{-\infty}^{\infty} \mathbf{x} \times (\mathbf{E} \times \mathbf{H})\, d^3x \tag{1.10}$$

Thus, with the value of l given in Eq. (1.2), the second term of the Poincaré integral is equal to the electromagnetic momentum and Eq. (1.4) gives the *constant total angular momentum* $\mathbf{J} = m\boldsymbol{\Lambda}$. The presence of a nonvanishing electromagnetic angular momentum is due to the axial character of the magnetic field created by a magnetic pole and acting on a scalar electric charge.

Let us add here a remark about symmetry (Lochak, 1997a, b): the Poincaré cone is enveloped by a vector $\mathbf{r}$, which is the *symmetry axis* of the system formed by the electric and the magnetic charges, and this axis rotates (with a constant angle $\boldsymbol{\Theta}'$) around the constant *angular momentum* $\mathbf{J} = \mathrm{m}\boldsymbol{\Lambda}$. But this is exactly the definition of the *Poinsot cone* associated with a symmetric top.

The identity of the Poincaré cone and the Poinsot cone of a symmetric top is not surprising because the system formed by electric and magnetic charges is axisymmetric and rotating around a fixed point with a constant total angular momentum, just like a top, but with a different radial motion because it is not rigid. Hence, the motion along the geodesic lines of the cone has nothing to do with a top.

Let us introduce the following definition, which has two obvious properties:

$$\mathbf{L} = \mathbf{r} \times \frac{d\mathbf{r}}{dt}; \quad L.\frac{\mathbf{r}}{r} = 0; \quad \boldsymbol{\Lambda}.\frac{\mathbf{r}}{r} = \lambda. \tag{1.11}$$

Figure 1.2 summarizes all of these points.

All the calculations and interpretations of Poincaré (1896) concerning an electric charge (a cathodic ray—that is. an electron) in the field of a magnetic pole are also right for a magnetic charge (a monopole) in the field of a coulombian electric pole. The cause of this is the symmetry of Coulomb's law between electricity and magnetism. We shall see later in this chapter, that this will be true in the case of our quantum equation for a magnetic monopole, which gives, at the classical limit, the Poincaré equation.

Consider another point: All the reasonings of Poincaré concerning the convergence phenomenon of cathodic rays observed by Birkeland are independent of the sign of magnetic charges, as Poincaré claimed, because his description depends only on the half-angle Θ' of the cone, which is defined by Eq. (1.8). Actually, by virtue of Eqs. (1.2) and (1.8), this angle depends on $V/\lambda = V\, mc/eg$, but an inversion of the sign of this ratio could be compensated by an inversion of time. Therefore, the crossing points between the trajectory and the angulat momentum would be same.

Nevertheless, the sign of charges appears in the rotation sense of the spiral trajectory of an electron along the cone, because the rotation of an electron (or of a monopole) around the cone is left or right according to the sign of V/λ. This is the unique echo of the opposite variances of electric and magnetic charges, which only quantum mechanics is able to describe clearly.

1.2 P. A. M. Dirac

Dirac (1931) asked the following question: "Why are all electric charges multiples of the same unit charge?". He considered exactly the same problem as Poincaré (the interaction between an electric charge and a fixed

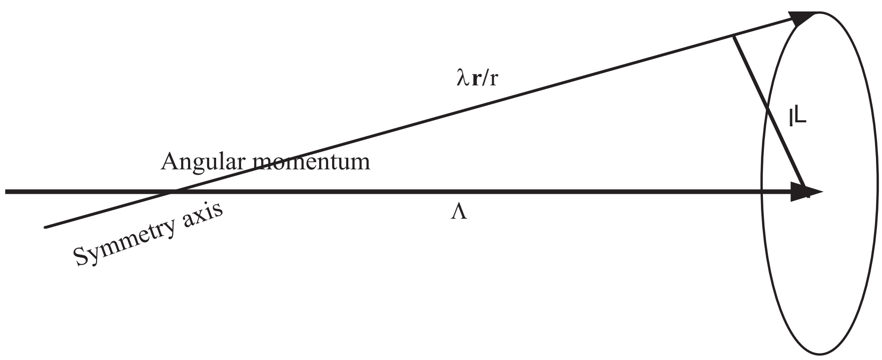

Figure 1.2 The generation of the Poincaré (or Poinsot) cone and the decomposition of the total momentum.

magnetic pole), but in quantum terms. This problem is exactly the same as the motion of a light magnetic monopole in the vicinity of a fixed electric charge. But there is a great difference: contrary to Poincaré, who knew the equation in classical mechanics, Dirac didn't know the quantum equation. We shall answer this question later in this discussion.

Here, we consider, as Dirac did, the motion of an electric charge e in the field of a fixed magnetic monopole with a charge g. The field $\mathbf{H}$ is thus defined by a vector potential $\mathbf{A}$ such that

$$curl\ \mathbf{A} = g\frac{\mathbf{r}}{r^3}. \tag{1.12}$$

It is clear that there is no continuous and uniform solution $\mathbf{A}$ of this differential equation because if we consider a surface S bounded by a loop Λ, we find according to the Stokes theorem:

$$\int_\Sigma \mathbf{H}.d\mathbf{S} = \int_\Sigma curl\mathbf{A}.d\mathbf{S} = \int_\Lambda \mathbf{A}.d\mathbf{l} = g\int_\Sigma \frac{\mathbf{r}}{r^3}.d\mathbf{S} = g\int_\Sigma d\Omega, \tag{1.13}$$

where d$\mathbf{S}$, d$\mathbf{l}$, and dΩ are elements of surface, length, and solid angle, respectively. Now, if the loop is shrunken to a point, while the pole remains inside the closed surface S, we get

$$\int_{\Lambda\to 0} \mathbf{A}.d\mathbf{l} = g\int_\Sigma d\Omega = 4\pi g. \tag{1.14}$$

This equality is impossible for a continuous potential $\mathbf{A}$ because the first integral vanishes. There must be a singular line around which the loop shrinks. Now, whatever the wave equation, the minimal coupling is given by a covariant derivative:

$$\nabla - i\frac{e}{\hbar c}\mathbf{A} \tag{1.15}$$

Dirac introduced into the wave function ψ a nonintegrable (nonunivalent) phase γ defining a new wave function:

$$\psi = e^{i\gamma}\psi. \tag{1.16}$$

If we apply the preceding operator [Eq. (1.15)], we know that the introduction of this phase γ is equivalent to the introduction of a new potential by a change of electromagnetic gauge:

$$\left(\nabla - i\frac{e}{\hbar c}\mathbf{A}\right)\psi = e^{i\gamma}\left(\nabla + i\nabla\gamma - i\frac{e}{\hbar c}\mathbf{A}\right)\psi. \tag{1.17}$$

We can identify the new potential with the gradient of γ, but the phase factor $e^{i\gamma}$ is admissible only if the variation of γ around a closed loop equals a multiple of 2π. Then, we must have

$$\frac{e}{\hbar c}\int_{\Lambda\to 0}\mathbf{A}\cdot d\mathbf{l}=\int_{\Lambda\to 0}\nabla\gamma\cdot d\mathbf{l}=(\Delta\gamma)_{loop}=2\pi n. \tag{1.18}$$

Comparing Eqs. (1.14) and (1.18), we find the *Dirac condition* between electric and magnetic charges:

$$\frac{eg}{\hbar c}=\frac{n}{2}. \tag{1.19}$$

It is interesting to confirm this result on a solution of Eq. (1.12). Dirac chose the following solution:

$$A_x=\frac{g}{r}\frac{-y}{r+z},\quad A_y=\frac{g}{r}\frac{x}{r+z},\quad A_z=0,\quad r=\sqrt{x^2+y^2+z^2}. \tag{1.20}$$

In polar coordinates, the solution is

$$x=r\sin\theta\cos\varphi,\quad y=r\sin\theta\sin\varphi,\quad z=r\cos\theta. \tag{1.21}$$

Eq. (1.20) becomes

$$A_x=\frac{g}{r}\tan\frac{\theta}{2}\sin\varphi,\quad A_x=\frac{g}{r}\tan\frac{\theta}{2}\cos\varphi,\quad A_z=0. \tag{1.22}$$

There is a nodal line that goes from $z=0$ to $z=\infty$ for $\theta=\pi$, and the Dirac condition is easily found if we compute the curvilinear integral [Eq. (1.18)] around this line for $\theta=\pi-\varepsilon$ and and $\varepsilon\to 0$. We must have

$$\frac{e}{\hbar c}=\int_{\Lambda\to 0}\mathbf{A}\cdot d\mathbf{l}=\frac{eg}{\hbar c}\int_{\theta=\pi-\varepsilon,\ \varepsilon\to 0}\frac{1}{r}\tan\frac{\theta}{2}r\sin\theta d\varphi=2\pi n. \tag{1.23}$$

Therefore,

$$\frac{eg}{\hbar c}\int_{\varepsilon\to 0}\frac{\sin\varepsilon}{\tan\frac{\varepsilon}{2}}d\varphi=\frac{eg}{\hbar c}2\times 2\pi=2\pi n. \tag{1.24}$$

Here, we see that the factor 2 comes from the factor $\varepsilon/2$ in the tangent, and we could conclude from that that it is related to the fact that the nodal line begins at $\mathbf{r}=0$. But this is wrong because the solution Eq. (1.20) or Eq. (1.22) chosen by Dirac depends on an arbitrary gauge; and in addition, his choice is not very good because *his potential has no definite parity*. Moreover, it must be stressed that with a *polar vector* **A**, the vector curl **A** is axial, so that Eq. (1.12) would be admissible only with a *pseudo-scalar constant g*, against which we have already objected. In the following discussion, we shall give the wave equation of a monopole in an electromagnetic field; our potential will not be **A**, but the pseudo-potential **B**, which will be a solution of the following equation (where e is the scalar electric charge):

$$curl\ \mathbf{B} = e\frac{\mathbf{r}}{r}. \tag{1.25}$$

B must be an axial vector, which is evident in Eq. (1.25), because *curl* **B** must be polar like **r**. *Mutatis mutandis*, Dirac's reasoning presented here will be true if we choose an axial solution of Eq. (1.25):

$$B_x = \frac{e}{r}\frac{yz}{x^2+y^2},\quad B_y = \frac{e}{r}\frac{-xz}{x^2+y^2},\quad B_z = 0,\quad r = \sqrt{x^2+y^2+z^2}. \tag{1.26}$$

This solution differs from the Dirac-like solution, which would be

$$B'_x = \frac{g}{r}\frac{-y}{r+z},\quad B'_y = \frac{g}{r}\frac{x}{r+z},\quad B'_z = 0. \tag{1.27}$$

In this, $\mathbf{B}'$ differs from **B** only by a gauge:

$$B - \mathbf{B}' = \nabla \arctan\frac{y}{x}. \tag{1.28}$$

In polar coordinates, Eq. (1.26) becomes

$$B_x = \frac{e}{r}\frac{\sin\varphi}{\tan\theta},\quad B_y = \frac{e}{r}\frac{-\cos\varphi}{\tan\theta},\quad B_z = 0. \tag{1.29}$$

Using Eq. (1.26) or (1.29) in Dirac's proof of the relation [Eq. (1.19)], the singular line goes from $-\infty$ to $+\infty$ instead of from 0 to $+\infty$ and the equality (1.24) becomes

$$2\times\frac{eg}{\hbar c}\int_{\varepsilon\to 0}\frac{\sin\varepsilon}{\tan\varepsilon}\,d\varphi = 2\times\frac{eg}{\hbar c}2\pi = 2\pi n. \tag{1.30}$$

This result gives Eq. (1.19) again, but now the factor 2 is no longer due to $\tan\ \varepsilon/2$, but due to the fact that the singular line pierces the sphere in *two points*. Therefore, the factor $n/2$ in the Dirac formula [Eq. (1.19)] was not at all related to the fact that the singular line began in r = 0. Nevertheless, this answer is not good either, and we shall prove further that *the factor $n/2$ is actually a consequence of the double connexity of the rotation group*.

According to Eq. (1.19), if we choose the charge e of the electron as a unit electric charge, the magnetic charge is quantized. For $n = 1$, we obtain the unit magnetic charge as a function of the electron charge and of the fine structure constant:

$$g_0 = \frac{\hbar c}{2e^2} e = \frac{e}{2\alpha} = \frac{137}{2} e = 68.5e. \tag{1.31}$$

This is a large charge, which is of the same order as the electric charge of a nucleus in the region of lantanides, beyond the middle of Dmitri Mendeleev's classification (137e is even beyond the classification). Nevertheless, this does not mean that such a monopole interacts with atoms as strongly as an electric charge of the same order. On the contrary, it must be stressed that all the experiments on monopoles are performed directly in the atmosphere of the laboratory, often at distances of several meters that cannot be crossed, for instance, by electrons. It can be undertood from the formula [Eq. (1.8)] of Poincaré, which shows that the total Lagrange moment increases with the Poincaré constant λ (proportional to the magnetic charge, as will be confirmed in quantum mechanics), the vertex angle Θ' of the cone decreases with the charge because it varies as λ^{-1}. Finally, it is the angle Θ' that gives the deviation of monopoles by an electric charge.

It is noteworthy that Dirac's condition [Eq. (1.19)] is based on general assumptions of quantum mechanics and electromagnetism, which is confirmed (despite some differences) by our equation (1.30). Nevertheless, we cannot forget that it was not systematically proved—and indeed, it has even been contradicted by many authors. For instance, we have already quoted the systematic, but contradictory, experiments of Mikhailov (Mikhailov, 1985, 1987, 1993). A paper of Price and colleagues (Price *et al.*, 1975) also identifies a track as being either one of a heavy nucleus, or of a monopole with a Dirac charge. And we remember the well-known measure of Blas Cabrera that gave the Dirac charge (Cabrera, 1982), but it was an "irreproducible result."

1.3 Pierre Curie

Among the symmetry laws stated by Pierre Curie, there is at least one that is well known and applied even by many who don't know that he was the first who stated it, at the beginning of his memoir, (Curie, 1894a,b)[1]:

[1] In Lochak (1997a, b), part of the Curie paper is given in a modern form, with consequences for the charges, electromagnetic potentials, and quantum mechanics that will be given later in the book.

> When some causes produce some effects, the elements of symmetry of causes must be found in the produced effects

Reciprocally, it is evident that

> If some effects reveal some asymmetry, this dissymmetry must be found in the causes that gave rise to these effects.

These laws are only two introductive lines of Curie's great memoir, which plays an essential role in what has followed it because it is essentially devoted to electromagnetism. But, as it was said in the Foreword, we shall follow this memoir only for a few pages, to give a foundation to some definitions. Then, we shall use more modern language and introduce some extensions.

The Spatial Symmetry of an Electric Field

Consider an electric field generated by two parallel coaxial circular plates of different metals. It has the symmetry of the cause: a revolution field around the axis, and every plane passing it will be a plane of symmetry. This is the symmetry of a *truncated cone,* but not yet of a *cone,* because the symmetry could be greater (cylindrical or spherical).

To find the exact symmetry, Curie takes a conductive, electrically charged sphere in a uniform electric field: "A force will act on the sphere in the direction of the field." The asymmetry of the effect must be found in the cause: the force exerted on the sphere has no symmetry axis normal to its direction, so the system sphere-field (the cause) no longer has such an axis. On the other hand, the sphere has infinite axes of symmetry, such that the cause of assymmetry is not in the sphere but in the field itself. Conclusion: the electric field cannot have a cylindrical or a spherical symmetry and it has the symmetry of a cone and the field may be represented by a polar vector (in $\mathbb{R}^3$). The same is true for a current or an electric polarization.

The Spatial Symmetry of a Magnetic Field

Consider the magnetic field generated at the center of a circular wire carrying a permanent current. The axis of the wire is an axis of *isotropy* and the plane of the circle is a *plane of symmetry*. Therefore, a magnetic field has a plane of symmetry normal to the direction of the field[2].

[2] This paradoxical symmetry is curiously represented on a painting of René Magritte: *La reproduction interdite*, which shows a man before a mirror who turns his back to the viewer. His image in the mirror turns his back too—just like a magnetic field!

On the other hand, the field has no binary normal axis, for the following reason. Take a rectilinear *conductive* bar moving normally along its length. This moving bar has a binary axis parallel to its velocity. Now, let us introduce a magnetic field normal to the bar and to the velocity: an electromotive force is generated in the bar, normal to it, and the binary axis disappears. Therefore, this axis must be absent from the cause, which means that a magnetic field has no orthogonal binary axis: it has the symmetry of a rotating cylinder. It may be represented by an axial vector (in $\mathbb{R}^3$). The same is true for a magnetic current or a magnetic polarization. Maxwell already knew that (Maxwell, 1873), without speaking of symmetry.

Now, from the reasoning of Pierre Curie, we can easily deduce the symmetry of charges, which is not given in his papers. Let us take the preceding circular electrically charged plates. A symmetry with respect to a parallel and equidistant plane will exchange between themselves the plates and the charges. Are the latter modified or not? We don't know it a priori, but we know that the electric field between the plates will be reversed. Thus, the electric charges are not changed: electric charges e are P-invariant. The conclusion would be the opposite for magnetic charges because in a similar experiment, we see that the reflected magnetic field is not changed. Therefore, magnetic charges g are P-reversed:

$$P: \mathbf{E} \to -\mathbf{E}; \quad \mathbf{H} \to \mathbf{H}; \quad e \to e; \quad g \to -g. \tag{1.32}$$

We shall see later in this chapter that these conclusions are confirmed in quantum mechanics for $\mathbf{E}$, $\mathbf{H}$, and e, but not for g, at least in this formulation. Such a change of the sign of a physical constant, like g, would be astonishing because it would signify that the constant g is a pseudo-scalar: a unique case in physics, *while all the other constants are true scalars* (see the Foreword). We shall see that this is not the case in quantum mechanics, but in the meanwhile, we shall keep the classical variance in another form.

Time Symmetry of Electromagnetic Fields

Curie didn't speak of time symmetry, which was not considered in his time. We shall start from the Lorentz force exerted by a field $\mathbf{E}$, $\mathbf{H}$ on an electric or a magnetic charge:[3]

$$\mathbf{F}_{elec} = e(\mathbf{E} + (1/c)\, \mathbf{v} \times \mathbf{H}); \quad \mathbf{F}_{magn} = g(\mathbf{H} - (1/c)\, \mathbf{v} \times \mathbf{E}) \tag{1.33}$$

[3] The formula for $\mathbf{F}_{magn}$ is easily found by applying the Lorentz transformation to the law $\mathbf{F} = g\mathbf{H}$ in the proper system.

These formulas cannot be contradicted by quantum mechanics because they must be found again at the geometrical optic limit. This is not enough to define variances, but it must be implicitly connected with them.

Now **F** is T-invariant (because $\mathbf{F} = m\gamma$), and **v** changes its sign with *t*, so we have, from Eq. (1.33):

$$T: e\mathbf{E} \to e\mathbf{E};\ \ e\mathbf{H} \to -e\mathbf{H};\ \ g\mathbf{H} \to g\mathbf{H};\ \ g\mathbf{E} \to -g\mathbf{E}. \tag{1.34}$$

Thus we have two possible variances:

$$\begin{array}{l} T_I: \mathbf{E} \to \mathbf{E},\ \ \mathbf{H} \to -\mathbf{H};\ \ e \to e, g \to -g \\ T_{II}: \mathbf{E} \to -\mathbf{E}, \mathbf{H} \to \mathbf{H};\ \ e \to -e, g \to g \end{array} \tag{1.35}$$

Such a case often happens: the electrodynamical phenomena only give a choice because they are able to define a link between the variances of several physical quantities, but not the variance of each quantity. It does not allow any possibility of an arbitrary choice[4]. Actually, in order to find the precise variances, we need some other phenomena, purely electric or purely magnetic (Curie, 1894a,b).

In this case, one can verify that to choose between the two possible laws [Eq. (1.35)], it is enough to find the variance of only one of the quantities **E**, **H**, e, g. We choose an *electrochemical phenomenon:* cathions heading to the anode with a current density: $\mathbf{J} = \rho\mathbf{v}$ (ρ is the density of cathions and **v** their velocity). Let us reverse the sign of time *t;* we do not know if the sign of charges is reversed, but in every case, the sign of ions and of the electrode remain opposite. Now, the sign of the velocity **v** is reversed; therefore, to conserve the density of current **J**, the sign of the electric charge must be reversed. Therefore, Law T_{II} is good and must be chosen.

Charge Conjugation and P, T, C Variances

In the forces [Eq. (1.33)], the fields **E** and **H** are exterior. Thus, they are independent of the charges *e* and *g* to which these fields are applied. But if a charge is reversed, the force is reversed, and thus we get

$$C: \mathbf{E} \to \mathbf{E},\ \ \mathbf{H} \to \mathbf{H},\ \ e \to -e,\ \ g \to -g. \tag{1.36}$$

Now, we can gather Eqs. (1.32), (1.35), and (1.36) into the **P**, **T**, **C** variances of fields and charges. As a result, we get the following table:

[4] For instance, such a choice is suggested in Jackson (1975, p. 249): "It is natural, convenient, and permissible to assume that charge is also a scalar under spatial inversion and even under time reversal." Of course, this is not an argument—and even if it were, it is wrong!

$$(I)\begin{bmatrix} P:\mathbf{E}\rightarrow-\mathbf{E}, & \mathbf{H}\rightarrow\mathbf{H}, & e\rightarrow e, & g\rightarrow-g \\ T:\mathbf{E}\rightarrow-\mathbf{E}, & \mathbf{H}\rightarrow\mathbf{H}, & e\rightarrow-e, & g\rightarrow g \\ C:\mathbf{E}\rightarrow\mathbf{E}, & \mathbf{H}\rightarrow\mathbf{H}, & e\rightarrow-e, & g\rightarrow-g \end{bmatrix}. \tag{1.37}$$

It must be emphasized that these P, T, C variances are directly deduced from experimental facts and from the laws of force [Eq. (1.33)], which are direct consequences of electromagnetism and relativity (and both are experimentally verified).

Symmetries of Electromagnetic Potentials

These symmetries are deduced from the definition of the electromagnetic fields **E** and **H**, which are related to the Lorentz potentials V and **A** or to the pseudopotentials W and **B**, which we cover later in this chapter[5]. W and **B** are the potentials "seen" by a magnetic pole, just as V and **A** are seen by an electric pole. Thus, we have two possible notations, for the electric case and for the magnetic case, respectively:

$$\mathbf{E}=-\nabla V-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\quad \mathbf{H}=\mathit{curl}\,\mathbf{A},\quad or:\mathbf{E}=\mathit{curl}\,\mathbf{B},\quad \mathbf{H}=\nabla W+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}. \tag{1.38}$$

From Eq. (1.37), we find the **P**, **T**, **C** variances of the potentials:

$$(II)\begin{bmatrix} P:\mathbf{A}\rightarrow-\mathbf{A}, & V\rightarrow V, & \mathbf{B}\rightarrow\mathbf{B}, & W\rightarrow-W, & e\rightarrow e, & g\rightarrow-g \\ T:\mathbf{A}\rightarrow\mathbf{A}, & V\rightarrow-V, & \mathbf{B}\rightarrow-\mathbf{B}, & W\rightarrow W, & e\rightarrow-e, & g\rightarrow g \\ C:\mathbf{A}\rightarrow\mathbf{A}, & V\rightarrow V, & \mathbf{B}\rightarrow\mathbf{B}, & W\rightarrow W, & e\rightarrow-e, & g\rightarrow-g \end{bmatrix}. \tag{1.39}$$

Let us make some remarks about these laws at this point:

a. The Lorentz transformation gathers the vector and scalar potentials $(\mathbf{A}, V)$ and $(\mathbf{B}, W)$, defined in $\mathbb{R}^3$, into two space-time quadrivectors:

$$A_\mu=(\mathbf{A}, iV);\quad iB_\mu=(\mathbf{B}, iW) \tag{1.40}$$

It is easy to introduce Eq. (1.39) into these expressions and to prove that A_μ and B_μ are polar and axial vectors in space and time, respectively (this is why there is an i before B_μ and not before A_μ).

b. The laws [Eq. (1.39)] give good $(P\ \ or\ \ T)$ variances $\mathbf{P}\rightarrow-\mathbf{P},\ \ E\rightarrow E$ for the Lagrange momenta:

[5] Here, we retain the notation **B**

$$\mathbf{P} = \mathbf{p} + \frac{e}{c}\mathbf{A}, \quad E = mc^2 + eV \text{ and } \mathbf{P} = \mathbf{p} + \frac{g}{c}\mathbf{B}, \quad E = mc^2 + gW \tag{1.41}$$

c_1. One can verify that the laws [Eq. (1.37)] ensures the invariance of the Maxwell equations:

$$\begin{gathered}
-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = curl\,\mathbf{E}, \quad \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = curl\,\mathbf{H}, \quad div\,\mathbf{H} = 0, \quad div\,\mathbf{E} = 0 \\
\mathbf{H} = curl\,\mathbf{A}, \quad \mathbf{E} = -\mathbf{grad}\,V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \quad \frac{1}{c}\frac{\partial V}{\partial t} + div\,\mathbf{A} = 0.
\end{gathered} \tag{1.42}$$

c_2. The laws [Eqs. (1.37) and (1.39)] ensure the invariance of the de Broglie equations of light, including the potentials (de Broglie, 1934):

$$\begin{gathered}
-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = curl\,\mathbf{E}, \quad \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = curl\,\mathbf{H} + k_0^2\mathbf{A} \\
div\,\mathbf{H} = 0, \quad div\,\mathbf{E} = -k_0^2 V \\
\mathbf{H} = curl\,\mathbf{A}, \quad \mathbf{E} = -\mathbf{grad}\,V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \quad \frac{1}{c}\frac{\partial V}{\partial t} + div\,\mathbf{A} = 0
\end{gathered} \tag{1.43}$$

c_3. Finally, the same laws [Eqs. (1.37) and (1.39)] ensure the invariance of *the equations of the magnetic photon* to which we already alluded. We shall return to it later, more precisely (Lochak, 1995a,b, 2003). The role of the potentials is played by the pseudopotentials as follows:

$$\begin{gathered}
-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = curl\,\mathbf{E} + k_0^2\mathbf{B}, \quad \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = curl\,\mathbf{H} \\
div\,\mathbf{H} = k_0^2 W, \quad div\,\mathbf{E} = 0 \\
\mathbf{H} = grad\,W + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \mathbf{E} = curl\,\mathbf{B}, \quad \frac{1}{c}\frac{\partial W}{\partial t} + div\,\mathbf{B} = 0.
\end{gathered} \tag{1.44}$$

The Curie symmetries, in quantum mechanics, will be given later, and that discussion will provide a stronger basis to the CPT symmetries, where differences with some accepted principles appear. An important result of Tables (I) and (II) above (which is absent from Curie's results, but which was deduced owing to his methods) is that the electric charge e is *P-invariant*, but *T-reversed,* and that the inverse is true for the magnetic charge g. And this will be true in quantum mechanics.

Let us make a conclusive remark concerning Maxwell and Curie. It is well known that, in all domains, important ideas may be lost for a long time. In the domain of symmetry, we face a phenomenon of this kind. Despite the fact that modern physics is dominated by symmetry, such great pioneers as Maxwell and Curie knew some results in electromagnetism that now have been more or less forgotten.

CHAPTER 2

A Wave Equation for a Leptonic Monopole, Dirac Representation

As was stated in the Foreword, our theory is not based on the Dirac works on monopoles, but on his famous theory of the electron. Our theory is based on two main points:

- The massless Dirac equation has a second gauge invariance, which defines a second electromagnetic interaction that obeys the laws of a magnetic monopole and the symmetry laws predicted by Pierre Curie. The monopole and the anti-monopole are chiral particles that are mirror images, as are the neutrino and the antineutrino, but here, it is true for magnetically charged particles, as it was predicted more than a century ago by Curie (Curie, 1894a, b, 1994).
- Contrary to other theories, our theory predicts that such a monopole is associated not with strong interactions, but with weak ones. And contrary to these other theories, the prediction is confirmed by experimentation.

There are naturally two paths for the theory, following either Dirac or Weyl. This chapter is devoted to the *Dirac representation*, while the next one will be devoted to the *Weyl representation*.

2.1 THE TWO GAUGE INVARIANCES OF DIRAC'S EQUATION

Consider the Dirac equation without an external field:

$$\gamma_\mu \partial_\mu \psi + \frac{m_0 c}{\hbar} \psi = 0, \tag{2.1}$$

where: $x_\mu = \{x_k; ict\}$ are the relativistic coordinates and the matrices γ_μ are expressed through the following Pauli s_k matrices:

$$\gamma_k = i\begin{pmatrix} 0 & s_k \\ -s_k & 0 \end{pmatrix}; \quad k = 1,2,3; \quad \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix};$$
$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{2.2}$$

Now let us define a general form of gauge transformation, where Γ is a constant Hermitian matrix and θ a constant phase:

$$\psi \to e^{i\Gamma\theta}\psi \tag{2.3}$$

At this point, introduce the gauge [Eq, (2.3)] into Eq. (2.1):

$$\left(\gamma_\mu e^{i\Gamma\theta}\gamma_\mu\right)\gamma_\mu\partial_\mu\psi + \frac{m_0 c}{\hbar}e^{i\Gamma\theta}\psi = 0. \tag{2.4}$$

Now, develop Γ on Clifford algebra as follows:

$$\Gamma = \Sigma_{N=1}^{16} a_N\Gamma_N; \quad \Gamma_N = I, \gamma_\mu, \gamma_{[\mu}\gamma_{\nu]}, \gamma_{[\lambda}\gamma_\mu\gamma_{\nu]}, \gamma_5, \tag{2.5}$$

and remember the relation $\gamma_\mu\Gamma_N\gamma_\mu = \pm\Gamma_N$ (Pauli, 1936), where $(\pm)$ depends on μ and N. We have

$$\gamma_\mu e^{i\Gamma\theta}\gamma_\mu = e^{i\theta \ \Sigma_{N=1}^{16} a_N\gamma_\mu\Gamma_N\gamma_\mu} = e^{i\theta \ \Sigma_{N=1}^{16} \pm a_N\Gamma_N}. \tag{2.6}$$

Eq. (2.1) remains invariant under the transformation [Eq, (2.4)] if Γ commute or anticommute with all the γ_μ; thus, we must have, in the last term of Eq. (2.6), either + or − before Γ_N for all γ_μ. We find $\Gamma = I$, for the plus sign, and $\Gamma = \gamma_5$ for the minus sign, and no other possibility. So

$$\Gamma = I \Rightarrow \psi \to e^{i\theta}\psi \ \ \textit{or} \quad \Gamma = \gamma_5 \Rightarrow \psi \to e^{i\gamma_5\theta}\psi. \tag{2.7}$$

The great difference is as follows:

- In the first case, $\Gamma = I$ commutes with the γ_μ: Eq. (2.4) is identical to Eq. (2.1), which is invariant under the transformation [Eq. (2.3)]. And we have defined the phase invariance, and $\psi \to e^{i\theta}\psi$ for any value of m_0, and we know that this ensures the conservation of charge.
- In the second case, $\Gamma = \gamma_5$ anticommutes with the γ_μ, so that the differential term in Eq. (2.1) has a minus sign in the exponential, while a plus sign remains in the exponential of the mass term. Therefore, the transformation $\psi \to e^{i\gamma_5\theta}\psi$ defines a gauge invariance only for a massless particle, at least for a linear equation (we shall see later in this chapter that things become different for nonlinear equations).

But, even with the nonlinear case, the symmetry has not broken. It has just become another symmetry: a *chiral symmetry,* which knows the difference between left and right, as was the case for magnetism (Maxwell, 1873; Curie, 1894a). We went from an electric particle, like an electron, to a magnetic monopole (Curie, 1894b, Lochak, 1985, 1995a,b, 2006).

It may be asserted that a monopole is not necessarily a super-heavy scalar: it can be a massless pseudoscalar. Indeed, we shall prove that the chiral invariance entails the conservation of magnetism, but with some important differences with respect to the conservation of electricity:

1. The conservation of magnetism is weaker than the conservation of electricity because its conservation is broken by the introduction of a linear mass term in the equation. Despite some analogies, the equations for an electron and a monopole are very different because of their different gauge laws.
2. The second difference is that, in (2.7): θ is a scalar phase for an electron and a pseudoscalar for a magnetic monopole. This is because γ_5 is a pseudoscalar operator, which implies two different mathematical worlds.

2.2 THE EQUATION OF THE ELECTRON

The Dirac equation of the electron ensues from the first transformation [Eq. (2.7)] generalized by a local gauge, in which the abstract angle θ is replaced by a physical angle φ with physical coefficients:

$$\psi \to e^{i\frac{e}{\hbar c}\varphi}\psi \tag{2.8}$$

So, introducing Eq. (2.8) in the differential term of Eq. (2.1), we find (up to the exponential factor)

$$\partial_\mu \psi \to e^{i\frac{e}{\hbar c}\varphi}\left(\partial_\mu \Psi + i\frac{e}{\hbar c}\partial_\mu \varphi\ \psi\right). \tag{2.9}$$

Now, we can generalize Eq. (2.8) by the adjunction of a potential:

$$\psi \to e^{i\frac{e}{\hbar c}\varphi}\psi;\quad \mathrm{A}_\mu \to \mathrm{A}_\mu + \partial_\mu \varphi. \tag{2.10}$$

Owing to Eqs. (2.9) and (2.10), Eq, (2.1) may be replaced by the following equation:

$$\gamma_\mu\left(\partial_\mu - i\frac{e}{\hbar c}\mathrm{A}_\mu\right)\psi + \frac{m_0 c}{\hbar}\psi = 0, \tag{2.11}$$

which is the Dirac equation of the electron in the presence of an electromagnetic field deriving from a Lorentz potential A_μ and which is invariant under the local gauge transformation [Eq. (2.10)]. The gauge transformation is local because it depends on space and time through an external electromagnetic field deriving from the potential A_μ.

Eq. (2.11) implicitly defines a *minimal coupling* and a *covariant derivative:*

$$\nabla_\mu = \partial_\mu - i\frac{e}{\hbar c}A_\mu \tag{2.12}$$

In the gauge transformation [Eq, (2.10)], the Lorentz potential A_μ is a polar vector and φ a scalar angle.

2.3 THE SECOND GAUGE, THE SECOND COVARIANT DERIVATIVE, AND THE EQUATION FOR A MAGNETIC MONOPOLE

Now, consider the Dirac equation [Eq. (2.1)] with $m_0 = 0$:

$$\gamma_\mu \partial_\mu \psi = 0. \tag{2.13}$$

This equation is invariant under both gauges [Eq. (2.7)]. We shall now examine the second one in the local case; i.e., with a pseudoscalar phase φ depending on the coordinates:

$$\psi \to e^{i\frac{g}{\hbar c}\gamma_5 \varphi}\psi. \tag{2.14}$$

Introducing the transformation [Eq. (2.14)] in Eq. (2.13), we find $\gamma_\mu\left(\partial_\mu + i\frac{g}{\hbar c}\gamma_5 \partial_\mu \varphi\right)\psi = 0$, which suggests a new minimal electromagnetic coupling by substituting the gradient of the pseudophase ϕ by the only possible potential, which is the pseudopotential defined in Eq. (1.40): $iB_\mu = (B, iW)$, from which we get a new covariant derivative:

$$\nabla_\mu = \partial_\mu - \frac{g}{\hbar c}\gamma_5 B_\mu. \tag{2.15}$$

In Eq. (2.15), i disppears because of the pseudoscalar character of γ_5. Finally, we find an new equation, which is the equation of a magnetic monopole (Lochak, 1983, 1984, 1985):

$$\gamma_\mu\left(\partial_\mu - \frac{g}{\hbar c}\gamma_5 B_\mu\right)\psi = 0. \tag{2.16}$$

This equation is relativistically invariant and gauge invariant, under the pseudoscalar transformation (with the same comment about i):

$$\psi \to e^{i\frac{g}{\hbar c}\gamma_5 \varphi}\psi; \quad B_\mu \to B_\mu + i\partial_\mu \varphi. \tag{2.17}$$

It will be proved later that the magnetic charge g is a scalar and not a pseudoscalar, which does not contradict Curie's laws because the pseudoscalar character of magnetism is not related to the number g but to the pseudoscalar magnetic charge operator $C = g\gamma_5$; i.e., to the pseudoscalar matrix γ_5. This matrix lies at the origin of the difference between classical and the quantum theories of magnetic monopoles.

2.4 THE DIRAC TENSORS AND THE "MAGIC ANGLE" *A* OF YVON-TAKABAYASI (FOR THE ELECTRIC AND THE MAGNETIC CASE)

It is known that in the Clifford basis [Eq. (2.5)], the Dirac spinor defines 16 bilinear tensorial quantities: a scalar, a polar vector, an antisymmetric tensor of rank 2, an antisymmetric tensor of rank 3 (an axial vector), and an antisymmetric tensor of rank 4 (a pseudoscalar):

$$\omega_1 = \overline{\psi}\psi;\ \ J_\mu = i\overline{\psi}\gamma_\mu\psi;\ \ M_{\mu\nu} = i\overline{\psi}\gamma_\mu\gamma_\nu\psi;\ \ \Sigma_\mu = i\overline{\psi}\gamma_\mu\gamma_5\psi;\ \ \omega_2 = -i\overline{\psi}\gamma_5\psi$$
$$\left(\overline{\psi} = \psi^+\gamma_4;\ \ \psi^+ = \psi\ \ h.c.\right). \tag{2.18}$$

If ω_1 and ω_2 do not vanish simultaneously, the Dirac spinor may be written as follows (Jacobi & Lochak, 1956a,b):

$$\psi = \rho\, e^{i\gamma_5 \mathbf{A}} U \psi_O. \tag{2.19}$$

where ρ = amplitude, U = general Lorentz transformation, ψ_O = constant spinor, and A = the *pseudoscalar angle* of Yvon-Takabayasi:

$$\rho = \sqrt{\omega_1^2 + \omega_2^2};\ \ A = \arctan\frac{\omega_2}{\omega_1}. \tag{2.20}$$

In Eq. (2.19), U is a product of six factors $e^{i\Gamma\,\vartheta}$, with three real Euler angles (rotations in R^3) and three imaginary angles (velocities in R^3). So we have seven angles in ψ: (1) three Euler angles, including the proper rotation angle φ, which gives a *half-scalar phase* $\varphi/2$ in the spinor Ψ, is conjugated by a *Poisson bracket* to the component J_4 of the *polar vector* J_μ; (2) the "imaginary three velocities," $i\frac{v_k}{c}$; (3) the *half-pseudoscalar* angle A conjugated to the Σ_4 component of the *axial vector* Σ_μ.

Both vectors J_μ and Σ_μ are defined in Eq, (2.18).

Angle A plays an important role in the Dirac theory of the electron because it appears in the tensor representation based on Eq, (2.18) (Takabayasi, 1957; Jacobi & Lochak, 1956a, b). Without A, the Dirac equation would be an equation of a classical relativistic spin fluid: the quantum properties are concentrated in the magic angle A, which appears in several tensorial equations deduced from the Dirac equation. The role played by the angle A in the theory of the magnetic monopole is even more fundamental.

For a discussion of all these questions, see Jakobi & Lochak (1956a, b), which give the classical-field Poisson brackets already noted in the Foreword and which are at the origin of the present theory of magnetic monopoles:

$$\left[\frac{\varphi}{2}, J_4\right] = \delta(r - r'); \quad \left[\frac{A}{2}, \Sigma_4\right] = \delta(r - r'). \tag{2.21}$$

In the electric case (Dirac theory of the electron), eJ_4 is a density of *electricity* and of *probability*, associated with the *phase invariance;* and the spatial part $e\mathbf{J}$ of eJ_μ is the current density of *electricity* or *probability*. As a result of the gauge invariance defined in Eq. (2.8), the Dirac equation [Eq. (2.11)] of the electron entails the conservation of electricity owing to the conservation of the polar vector eJ_μ:

$$\partial_\mu\left(eJ_\mu\right) = \left(\partial_\mu ie\overline{\psi}\gamma_\mu\psi\right) = 0. \tag{2.22}$$

In the magnetic case (equation of the monopole), the *polar electric current density* eJ_μ is replaced by *the axial magnetic current density* $K_\mu = g\Sigma_\mu$. The time and space components (K_4 and $\mathbf{K}$), of K_μ will be the *densities of magnetic charge and of magnetic current,* respectively. As a consequence of the *gauge invariance* [Eq. (2.17)], the equation of the monopole [Eq. (2.16)] entails the conservation of magnetism through the conservation of the axial vector density K_μ:

$$\partial_\mu K_\mu = 0; \quad \left\{K_\mu = g\Sigma_\mu = g\left(i\overline{\psi}\gamma_\mu\gamma_5\psi\right)\right\}. \tag{2.23}$$

Now, it must be noticed that owing to the expressions of J_μ and Σ_μ in terms of Ψ [Eq. (2.18)], one can prove the following:

1. J_μ is *polar*, Σ_μ is an *axial vector* or a *pseudovector:* the definition [Eq, (2.23)] shows that Σ_μ is the dual of a completely antisymmetric tensor of the third rank: $\{i\gamma_2\gamma_3\gamma_4,\ i\gamma_3\gamma_1\gamma_4, i\gamma_1\gamma_2\gamma_4, \gamma_1\gamma_2\gamma_3\}$.
2. J_μ is *timelike* and Σ_μ is *spacelike,* by virtue of the Darwin—de Broglie equalities:

$$-J_\mu J_\mu = \Sigma_\mu\Sigma_\mu = \omega_1^2 + \omega_2^2; \ J_\mu\Sigma_\mu = 0. \tag{2.24}$$

The expression Σ_μ for the magnetic current was already suggested by Salam (1966) for symmetry. But in this case, this is not an a priori definition–rather, it is a consequence of the wave equation and of the second gauge condition [Eq. (2.17)].

Here, it must be stressed that Dirac's theory defines only two vectors, without derivatives: J_μ and Σ_μ. Because J_μ is polar and *timelike,* it may be interpreted as a current density of electricity and probability. Because Σ_μ is axial, it may be interpreted as a current density of magnetism: this double coincidence is a remarkable example of harmony between physics and mathematics. We shall see a little later, in the discussion of the Weyl representation, that the spacelike character of Σ_μ is by no means an objection against its interpretation as a current: it will still reinforce this mathematical harmony[6].

2.5 *P, T, C* SYMMETRIES. PROPERTIES OF THE ANGLE *A* (NOT TO BE CONFUSED WITH THE LORENTZ POTENTIAL A)

Even though we shall be discussing the transformation of the wave function later in this chapter, it is interesting to say here that according to our theory, the *P, T, C* invariances are in perfect accordance with Curie's laws.

In the electric case, the correct transformations given by the P, T, C invariances of the Dirac equation [Eq. (2.11)] are the following, where A_k and A_4 are the Lorentz potentials (Lochak, 1997a, b):

$$\begin{aligned} &P : e \to e;\ \ x_k \to -x_k;\ \ x_4 \to x_4;\ \ \psi \to \gamma_4 \psi \\ &\ A_k \to -A_k;\ \ A_4 \to A_4 \\ &T : e \to -e;\ \ x_k \to x_k;\ \ x_4 \to -x_4;\ \ \psi \to -i\gamma_3\gamma_1\psi^* \\ &\ A_k \to A_k;\ \ A_4 \to -A_4 \\ &C : e \to -e;\ \ \psi \to \gamma_2 \psi^* \end{aligned} \tag{2.25}$$

where P and C are the Racah transformations (Racah, 1937), but T is not, because, as we have seen in Chapter 1, the electric charge e is reversed by the T transformation, which leads to the antilinear wave transformation $\Psi \to -i\gamma_3\gamma_1\Psi^*$, often known as *weak time reversal*[7].

We shall now adopt this law as the true time reversal. This is always true, including in the case of a magnetic charge, because one can easily prove that the P, T, C invariances of the monopole equation [Eq, (2.16)] are given by

[6] For a long time, Σ_μ was considered the spin vector, because its space components appeared in the Dirac expression of total angular momentum: $-i(x_j\partial_k - x_k\partial_j) + \sigma_k$ $(i, j, k = \textit{circular permutation}, \sigma_k = \text{“}\textit{spin matrices}\text{”})$.

[7] The Racah T transformation, $\psi \to \gamma_1\gamma_2\gamma_3\psi$, contradicts the transformation $e \to -e$.

$$
\begin{aligned}
&P: g \to g; \quad x_k \to -x_k; \quad x_4 \to x_4; \quad \psi \to \gamma_4 \psi \\
&\;\; B_k \to B_k; \quad B_4 \to -B_4 \\
&T: g \to g; \quad x_k \to x_k; \quad x_4 \to -x_4; \quad \psi \to -i\gamma_3\gamma_1\psi^* \\
&\;\; B_k \to B_k; \quad B_4 \to -B_4 \\
&C: g \to g; \quad \psi \to \gamma_2 \psi^*
\end{aligned}
\tag{2.26}
$$

In Eq. (2.25), contrary to Eqs. (1.37) and (1.39), the magnetic charge g is invariant in the three transformations P, T, C.

The pseudoscalar character of magnetism is not given by the constant g, but by the *charge-operator* $g\gamma_5$ which lies at the origin of all the differences between the classical and quantum theories of magnetic monopoles. Now it may be shown that $\omega_1 = \overline{\psi}\psi$ is really a scalar, and $\omega_2 = -i\overline{\psi}\gamma_5\psi$ a pseudoscalar, as a consequence of the P, T, C transformations of the spinor ψ given in Eqs. (2.25) and (2.26) and applied to the formulas of these quantities given in the list [Eq. (2.18)]. An elementary calculation gives

$$
\begin{aligned}
&P: \omega_1 \to \omega_1; \quad \omega_2 \to -\omega_2; \quad T: \omega_1 \to \omega_1; \quad \omega_2 \to -\omega_2; \\
&C: \omega_1 \to -\omega_1; \quad \omega_2 \to -\omega_2.
\end{aligned}
\tag{2.27}
$$

Therefore, ω_1 represents P and T invariants, and ω_2 represents P and T pseudoinvariants. And they are both reversed by C so that they are not PTC invariants. On the contrary, it is easy to prove that they are both relativistic invariants.

The definition [Eq. (2.19)] of the angle **A** shows, owing to Eq. (2.26), that

1. The angle A is a relativistic invariant.

2. The sign of A is reversed by P **and by** T so that A is a relativistic pseudoscalar (in $\mathbb{R}^4$).

3. The angle A is C invariant. Therefore, A is PTC invariant.

Now a geometrical interpretation of the chiral gauge may be given. We shall first define a chiral plane, in which we consider a vector (ω_1, ω_2): actually, (ω_2) is reversed when x or t is reversed. By virtue of Eq. (2.20), the angle A is a pseudoangle, so that the vector with coordinates (ω_1, ω_2) may be defined by

$$
\omega_1 = \rho \cos A; \quad \omega_2 = \rho \sin A. \tag{2.28}
$$

Now, consider a rotation θ in the plane (ω_1, ω_2), defined by a rotation $\theta/2$ of a spinor:

$$
\psi' \to e^{i\gamma_5/2}\psi. \tag{2.29}
$$

Using the definition [Eq. (2.18)] of $\omega_1 = \overline{\psi}\psi$ and $\omega_2 = -i\overline{\psi}\gamma_5\psi$, we find from Eq. (2.28) the rotation of the vector (ω_1, ω_2):

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \Rightarrow A' = A + \theta. \tag{2.30}$$

Therefore, the second gauge invariance [Eq. (2.29)] is a rotation, just like the first one, but it is a rotation in the chiral plane, not in the physical space.

Now the quantity ρ will be called the *principal chiral invariant*. The rotation angle $\theta/2$ of the spinor is equal to half the rotation angle θ of a vector in the chiral plane, in accordance with the spinor geometry.

Finally, as we have seen, according to Eq. (2.26), the charge conjugation does not change the sign of the magnetic constant of charge g, which means that two monopoles with opposite constants g are not charge-conjugated: we shall see that a change of g to $-g$ signifies a change of the vertex angle of the Poincaré cone. In the next chapter, we shall see what charge conjugation means in the magnetic case, but it may be stated here that two conjugated monopoles have the same charge constant g.

We cannot create or annihilate pairs of monopoles with charges g *and* $-g$, as was the case for electric charges e *and* $-e$. As a result, there is no danger of an infinite polarization of the vacuum with such zero mass monopoles. Moreover, one has not to invoke great masses to explain the rarity of monopoles or the difficulty of observing them. There are other reasons for this, which will be explored later in this book.

CHAPTER 3

The Wave Equation in the Weyl Representation. The Interaction Between a Monopole and an Electric Coulombian Pole. Dirac Formula. Geometrical Optics. Back to Poincaré

This chapter will explore the same monopole equation as Chapter 2, but for the *Weyl representation*.

3.1 THE WEYL REPRESENTATION

We shall define the Weyl representation by the following transformation (Lochak, 1983, 2006), which divides the wave function ψ into the two-component spinors ξ and η:

$$\Psi \to U\Psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}; \quad U = U^{-1} = \frac{1}{\sqrt{2}}(\gamma_4 + \gamma_5); \quad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$
$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

The matrix γ_5 and the magnetic charge operator C are diagonalized:

$$UBU^{-1} = Ug\gamma_5 U^{-1} = g\gamma_4 = \begin{pmatrix} g & 0 \\ 0 & -g \end{pmatrix}. \tag{3.2}$$

Eqs. (3.1) and (3.2) show that ξ and η are eigenstates of B, with eigenvalues g and $-g$:

$$UBU^{-1}\begin{pmatrix} \xi \\ 0 \end{pmatrix} = g\begin{pmatrix} \xi \\ 0 \end{pmatrix}; \quad UBU^{-1}\begin{pmatrix} 0 \\ \eta \end{pmatrix} = -g\begin{pmatrix} 0 \\ \eta \end{pmatrix}. \tag{3.3}$$

Owing to Eqs. (3.1) and (1.40), Eq. (2.16) splits into a pair of uncoupled two-component equations in ξ and η, corresponding to the opposite eigenvalues of B:

$$\begin{aligned} &\left[\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{s}.\nabla - i\frac{g}{\hbar c}(W + \mathbf{s}.\mathbf{B})\right]\xi = 0 \\ &\left[\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{s}.\nabla + i\frac{g}{\hbar c}(W - \mathbf{s}.\mathbf{B})\right]\eta = 0. \end{aligned} \tag{3.4}$$

The P, T, C symmetries [Eq. (2.25)] take the form:

$$\begin{aligned} &P: g \to g; \quad x_k \to -x_k; \quad t \to t; \quad B_k \to B_k; \quad W \to -W; \quad \xi \leftrightarrow \eta \\ &T: g \to g; \quad x_k \to x_k; \quad t \to -t; \quad B_k \to -B_k; \quad W \to W; \quad \xi \to s_2\xi^*; \eta \to s_2\eta^* \\ &C: g \to g; \quad \xi \to -is_2\eta^*; \quad \eta \to is_2\xi^*. \end{aligned} \tag{3.5}$$

P and T exchange Eq. (3.4) between themselves.

Thus, we have a pair of *charge conjugated* particles—a *monopole* and an *antimonopole*—with the same charge constant g and *opposite helicities*. They are defined by the operator C, which shows that our monopole is a magnetically *excited neutrino because* Eq. (3.4) reduces to a pair of two-component neutrino equations if $g = 0$.

Eq. (3.4) is invariant under the following gauge transformation (with opposite signs of the phase of ξ and η, which is nothing but the Weyl representation of the gauge transformation [Eq. (2.16)]:

$$\xi \to \exp\left(i\frac{g}{\hbar c}\phi\right)\xi;\, \eta \to \exp\left(-i\frac{g}{\hbar c}\phi\right)\eta;\ W \to W + \frac{1}{c}\frac{\partial\phi}{\partial t};\ \mathbf{B} \to \mathbf{B} - \nabla\phi. \tag{3.6}$$

3.2 CHIRAL CURRENTS

The gauge [Eq. (3.6)] entails, for Eq. (3.4), the following *conservation laws*:

$$\frac{1}{c}\frac{\partial\left(\xi^{+}\xi\right)}{\partial t} - \nabla\xi^{+}\mathbf{s}\xi = 0;\ \frac{1}{c}\frac{\partial\left(\eta^{+}\eta\right)}{\partial t} + \nabla\eta^{+}\mathbf{s}\eta = 0. \tag{3.7}$$

Thus, we have *two currents* with several important properties. They are *isotropic* and *chiral,* and they *exchange between themselves by parity*:

$$\begin{aligned} &X_\mu = \left(\xi^{+}\xi, -\xi^{+}\mathbf{s}\xi\right);\ Y_\mu = \left(\eta^{+}\eta, \eta^{+}\mathbf{s}\eta\right); X_\mu X_\mu = 0; \\ &Y_\mu Y_\mu = 0;\ P \Rightarrow X_\mu \leftrightarrow Y_\mu. \end{aligned} \tag{3.8}$$

Owing to Eq. (3.1), we find a decomposition of the polar and axial vectors, as defined in Eq. (2.17):

$$J_\mu = X_\mu + Y_\mu;\ \Sigma_\mu = X_\mu - Y_\mu. \tag{3.9}$$

The chiral currents X_μ and Y_μ may be considered even more fundamental than electric and magnetic currents. We already know the relations [Eq. (2.23)], and it is easy to prove, by using Eqs. (2.17) and (3.1), that

$$\begin{aligned} &\omega_1 = \xi^{+}\eta + \eta^{+}\xi;\ \ \omega_2 = i\left(\xi^{+}\eta - \eta^{+}\xi\right); \\ &\rho^2 = \omega_1^2 + \omega_2^2 = 4\left(\xi^{+}\eta\right)\left(\eta^{+}\xi\right). \end{aligned} \tag{3.10}$$

It was noted in Chapter 2 that a consequence of Eq. (2.23) is that J_μ is timelike and Σ_μ is spacelike. Owing to Eq. (3.9), we can add that the fact that one of the vectors (J_μ, Σ_μ) is timelike and the other spacelike is a trivial property of the addition and subtraction of isotropic vectors. And if J_μ is *precisely* spacelike and Σ_μ spacelike, this is due to the + sign of $(\omega_1^2 + \omega_2^2)$ in Eq. (2.23).

Therefore, our magnetic current, $K_\mu = g\,\Sigma_\mu$, may be spacelike because the true magnetic currents are the isotropic currents $g\,X_\mu$ and $-g\,Y_\mu$, corresponding to the spinor states ξ and η. The pseudovector K_μ **is only their difference,** so it has no reason to be spacelike or timelike. Therefore, the relativistic type of the magnetic current K_μ has no importance; on the contrary, the fact that J_μ is timelike is very important because owing to this property, J_μ may be interpreted as a current density of probability or electricity. Moreover, J_μ is a polar vector, which is necessary for a current of probability or electricity, while Σ_μ is a pseudovector, as a magnetic current must be (Curie, 1894a,b). We have already noted this beautiful example of harmony between physics and mathematics.

3.3 A REMARK ABOUT THE DIRAC THEORY OF THE ELECTRON

The equations of current continuity [Eq. (3.7)] were deduced from the Weyl representation [Eq. (3.4)] of the equation of the magnetic monopole [Eq. (2.15)]. It is interesting to compare that result with the Weyl representation of the equation of the electron, applying the transformation [Eq. (3.1)] to Eq. (2.10) instead of (2.15).

Taking into account the equality $A_\mu = (\mathbf{A}, iV)$, we find a system that is the analog of Eq. (3.4) but equivalent to the Dirac equation:

$$\begin{aligned}
&\left[\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{s}.\nabla + i\frac{e}{\hbar c}(V + \mathbf{s}.\mathbf{A})\right]\xi + i\frac{m_0}{\hbar c}\eta = 0 \\
&\left[\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{s}.\nabla + i\frac{g}{\hbar c}(V - \mathbf{s}.\mathbf{A})\right]\eta + i\frac{m_0}{\hbar c}\xi = 0.
\end{aligned} \tag{3.11}$$

Let us notice some points here:

1. Eqs. (3.4) and (3.11) has the same differential part.
2. Thus, in the massless case, we find in both systems two separate equations for the chiral components ξ, η (i.e., for opposite helicities). It is known that in Eq. (2.15) or (3.4), the condition $m_0 = O$ is a consequence of the chiral gauge invariance $\Psi \to \exp\left(i\frac{g}{\hbar c}\gamma_5\phi\right)\Psi$. Nevertheless, the fact that chiral components obey separate equations [Eqs. (3.4) and (3.11)] depends only on the zero mass, whatever the reason for this zero mass may be and whatever the charge of the particle is.

3. Now, from the Dirac system [Eq. (3.11)], with $m_0 \neq O$ in the case of an electric interaction, it is easy to deduce the evolution law of isotropic currents:

$$\begin{aligned}
&\frac{1}{c}\frac{\partial(\xi^+\xi)}{\partial t} - \nabla\xi^+\mathbf{s}\xi + i\frac{m_0c}{\hbar}(\xi^+\eta - \eta^+\xi) = 0 \\
&\frac{1}{c}\frac{\partial(\eta^+\eta)}{\partial t} + \nabla\eta^+\mathbf{s}\eta - i\frac{m_0c}{\hbar}(\xi^+\eta - \eta^+\xi) = 0.
\end{aligned} \tag{3.12}$$

We see here that the law does not depend explicitly on electromagnetic interaction. The difference between electricity and magnetism appears in the presence of a mass term only in the case of the electron, a term that is excluded by the chiral gauge in the case of a magnetic monopole. The chiral gauge invariance is the true difference between the two theories because it introduces, in Eq. (3.4), the magnetic interaction that is responsible for new forces.

Taking Eqs. (3.8) and (3.10) into account, we find the following laws, which mean that the Dirac pseudoinvariant ω_2 is the source of chiral isotropic currents:

$$\partial_\mu X_\mu + i\frac{m_0c}{\hbar}\omega_2 = 0; \quad \partial_\mu Y_\mu - i\frac{m_0c}{\hbar}\omega_2 = 0. \tag{3.13}$$

Adding and subtracting these equalities, we find two well-known laws:

$$\partial_\mu J_\mu = 0; \quad \partial_\mu \Sigma_\mu + 2i\frac{m_0c}{\hbar}\omega_2 = 0. \tag{3.14}$$

Eq. (3.13) expresses the conservation of electricity and probability, by the Dirac equation. Eq. (3.14) is called, in Dirac's theory, the *Uhlenbeck and Laporte equality*. Starting from our theory of the leptonic monopole, we see that Eq. (3.13) governs the evolution of the left and right isotropic currents generated by the Dirac pseudoinvariant, which implies that Eq. (3.14) of *Uhlenbeck and Laporte* governs their difference, Σ_μ.

At this point, it is important to notice a fundamental difference between electricity and magnetism; in the Dirac equation, there is conservation of neither isotropic currents X_μ and Y_μ, nor of their difference $\Sigma_\mu = X_\mu - Y_\mu$. As a result, there is no conservation of magnetism; on the contrary, the sum $J_\mu = X_\mu + Y_\mu$ is conserved, and this is the conservation of electricity. The latter is related only to the presence of a mass term, but the following must be underlined:

1. We cannot add to Eq. (3.11) a magnetic interaction because it would be contrary to the presence of the mass term.

2. We cannot introduce into Eq. (3.4) an electric interaction because there is no Dirac massless electron, which would not admit quantized states and would provoke difficulties with the creation and annihilation of pairs.

Therefore, "leptonic dyons" carrying both electric and magnetic charges cannot exist.

3.4 THE INTERACTION BETWEEN A MONOPOLE AND AN ELECTRIC COULOMBIAN POLE (ANGULAR FUNCTIONS)

To solve the problem of a central field, we must introduce W = 0 and either Eq. (1.26) or (1.29) of **B** in the chiral equations [Eq. (3.4)]. The Poincaré integral [Eq. (1.4)] takes, in the quantum case, the expressions given next, in Eq. (3.15), for the left and right monopole. For the time being, we shall admit that result without proof, which will be given in the next chapter, in a more general case:

$$\mathbf{J}_\xi = \hbar\left[\mathbf{r}\times(-i\nabla + D\,\mathbf{B}) + D\,\mathbf{r} + \frac{1}{2}\mathbf{s}\right]$$
$$\mathbf{J}_\eta = \hbar\left[\mathbf{r}\times(-i\nabla - D\,\mathbf{B}) - D\,\mathbf{r} + \frac{1}{2}\mathbf{s}\right]. \tag{3.15}$$

$\mathbf{J}_\xi$ and $\mathbf{J}_\eta$ differ only by the sign of D; i.e., by the sign of the eigenvalues of the charge operator $C = g\gamma_5$, defined in Eq. (2.16). The notations are

$$D = \frac{eg}{\hbar c}, \quad \mathbf{B} = eB, \quad \widehat{\mathbf{r}} = \frac{\mathbf{r}}{r}, \tag{3.16}$$

where D is the *Dirac number,* which we already know from the *Dirac condition* [Eq. (1.19)]—the last will be found below a new form; and B is the pseudopotential [Eq. (1.26) or (1.29)].

As was said previously, the proof that $\mathbf{J}_\xi$ and $\mathbf{J}_\eta$ are first integrals of Eq. (3.4) will be given in the next chapter. But for now, it is easy to show that the components of $\mathbf{J}$ obey the relations:

$$[\mathbf{J}_2,\mathbf{J}_3] = i\hbar\,\mathbf{J}_1,\ [\mathbf{J}_3,\mathbf{J}_1] = i\hbar\,\mathbf{J}_2,\ [\mathbf{J}_1,\mathbf{J}_2] = i\hbar\,\mathbf{J}_3 \tag{3.17}$$

Here, we shall only find their proper states, restricting our demonstration to the plus sign of *D;* i.e., to the left monopole—the first expression in Eq. (3.4)—dropping the index ξ.

Now, let us write $\mathbf{J}$ as

$$\mathbf{J} = \hbar\left[\boldsymbol{\Lambda} + \frac{1}{2}\mathbf{s}\right], \quad \boldsymbol{\Lambda} = \mathbf{r} \times (-i\nabla + D\mathbf{B}) + D\widehat{\mathbf{r}}. \tag{3.18}$$

One can see that $\hbar\boldsymbol{\Lambda}$ is *the quantum form of the Poincaré first integral* [Eq. (1.4)] (Poincaré, 1896). $\mathbf{J}$ is the sum of the quantum form $\hbar\boldsymbol{\Lambda}$ of the first integral and of the spin operator $\hbar\mathbf{s}$: $\mathbf{J}$ is the *total quantum angular momentum* of the monopole in an electric coulombian field, the generalization of the classical quantity. Of course, the components of $\hbar\boldsymbol{\Lambda}$ obey the same relations [Eq. (3.16)] as the components of $\mathbf{J}$ because $\boldsymbol{\Lambda}$ commutes with $\mathbf{s}$.

In polar angles, from the definition [Eq. (3.18)] of $\boldsymbol{\Lambda}$ and from the polar form [Eq. (1.29)] of B, we find the following:

$$\begin{aligned} \Lambda^+ &= \Lambda_1 + i\Lambda_2 = e^{i\varphi}\left(i\cot\theta \frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\theta} + \frac{D}{\sin\theta}\right) \\ \Lambda^- &= \Lambda_1 - i\Lambda_2 = e^{-i\varphi}\left(i\cot\theta \frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta} + \frac{D}{\sin\theta}\right) \\ \Lambda_3 &= -i\frac{\partial}{\partial\varphi}. \end{aligned} \tag{3.19}$$

Let us note that, owing to our choice [Eq. (1.26)] for the electromagnetic gauge, there is no additional term in Λ_3, contrary to the findings of Wu and Yang (1975, 1976). Now, we need the eigenstates $Z(\theta, \varphi)$ of Λ^2 and Λ_3. By virtue of Eq. (3.16), the eigenvalue equations of Λ must be

$$\begin{aligned} &\Lambda^2 Z = j(j+1)Z, \quad \Lambda_3 Z = mZ, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, 3, \ldots, \\ &m = -j, -j+1, \ldots, j-1. \end{aligned} \tag{3.20}$$

To simplify the calculation of Z(θ, φ), we shall introduce a new angle χ, the meaning of which will be given shortly. We write

$$\mathcal{D}(\theta, \varphi, \chi) = e^{iD\chi} Z(\theta, \varphi), \tag{3.21}$$

where the functions $\mathcal{D}(\theta, \varphi, \chi)$ are the eigenstates of operators R_k, which are easily derived from Eq. (3.19):

$$\begin{aligned} R^+ &= R_1 + iR_2 = e^{i\varphi}\left(i\cot\theta \frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\chi}\right) \\ R^- &= R_1 - iR_2 = e^{-i\varphi}\left(i\cot\theta \frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\chi}\right) \\ R_3 &= -i\frac{\partial}{\partial\varphi} \end{aligned} \tag{3.22}$$

Obviously, the eigenvalues are the same as those of Z:

$$R^2 Z = j(j+1)Z, \quad R_3 Z = mZ, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, 3, \ldots, \\ m = -j, -j+1, \ldots, j-1. \tag{3.23}$$

The operators R_k are well known: they are the *infinitesimal operators of the rotation group written in* the *fixed referential*. The angles θ, φ, χ are the Euler angles of nutation, precession, and proper rotation. The role of the rotation group is not surprising because of the *spherical symmetry* of the system constituted by a monopole in a central electric field.

Our eigenfunction problem is thus trivially solved: instead of the cumbersome calculations of monopole harmonics that do not exist, we see, under the simple assumption of continuity of the wave functions with respect to the rotation group, that the angular functions are the *generalized spherical functions;* i.e., the matrix elements of the irreducible unitary representations of the rotation group (Gelfand, Minlos, & Shapiro, 1963; Lochak, 1959). These functions are also the eigenfunctions of the symmetrical top. This coincidence was noticed by Tamm in 1931 without explanation, but here the explanation is evident because we already know the analogy between a symmetrical top and a monopole in a central field. The eigenstates of R^2 and R_3 are

$$\mathcal{D}_j^{m',m}(\theta, \varphi, \chi) = e^{i(m\varphi + m'\chi)} d_j^{m',m}(\theta)$$
$$d_j^{m',m}(\theta) = N(1-u)^{\frac{-(m-m')}{2}}(1+u)^{\frac{-(m+m')}{2}}\left(\frac{d}{du}\right)^{j-m}\left[(1-u)^{j-m'}(1+u)^{j+m'}\right]$$
$$u = \cos\theta, \quad N = \frac{(-1)^{j-m} i^{m-m'}}{2^j}\left(\frac{(j+m)!}{(j-m)!(j-m')!(j+m')!}\right)^{1/2}$$
$$j = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \quad m, m' = -j, -j+1, \ldots, j-1, j. \tag{3.24}$$

The normalization factor N is so defined that rows and columns of the unitary (2j + 1) matrix of the representation $\mathcal{D}_j$ are normed to unity. To normalize the quantum states, we must take the factor Z in Eq. (3.21) in the form

$$Z_j^{m',m}(\theta, \varphi) = \sqrt{2j+1}\ \mathcal{D}_j^{m',m}(\theta, \varphi, 0). \tag{3.25}$$

The proper rotation angle χ does not appear in $Z_j^{m',m}(\theta,\varphi)$: it appears only in the phase $e^{iD\chi}$ (D = Dirac number) because the monopole was implicitly supposed to be a point contrary to the symmetric top that has a spatial extension. Nevertheless, there is a projection (different from zero), of the orbital angular momentum on the symmetry axis, due to the chirality of the magnetic charge. The eigenvalue associated to the projection is the quantum number m'. The crucial point is that, if we compare Eqs. (3.21) and (3.24), we see that the eigenvalue $\hbar m'$ of the projection must be equal to the Dirac number D.

The quantization of the Dirac number D, thus is a consequence of the continuity of the wave function, on the rotation group.

Taking into account Eq. (3.23), we find

$$D = \frac{eg}{\hbar c} = m' = -j, -j+1, \ldots, j-1, j;\;\; j = \frac{n}{2}. \tag{3.26}$$

Taking into account the definition [Eq. (3.16)], we see that the equality [Eq. (3.26)] is a new and more precise form of the Dirac condition [Eq. (1.19)]. In this new formula, the integer (or half-integer) m' is not an arbitrtay number as it was in the Dirac formula. Rather, m' is now defined by the projection of the angular momentum of the whole physical system on the symmetry axis passing through the two charges.

The condition [Eq. (3.26)], which implies the Dirac condition [Eq. (1.19)], appears as a consequence of the spherical symmetry of the system and of the continuity of the wave functions with respect to the rotation group. It is justified by a dynamical argument, not only formally derived.

As was already stated, the factor of one-half has nothing to do with the strings beginning at the origin: it is a consequence of the double connexity of the rotation group that appears in the presence of half-integers in the representations of the group, and thus in the corresponding values of j and m'. Let us draw attention to, concerning these questions, an important work of T. W. Barrett in which the role of the rotation group in electromagnetic field theories is extensively developed (Barrett, 1989).

Now, owing to Eqs. (3.16) and (3.26), we can define the values of the magnetic charges as functions of the charge of the electron, the Planck constant, and the velocity of light because the value g_0 of the fundamental magnetic charge is given for $n = 1$ by Eq. (3.26), and the others are multiples of this value:

$$g_0 = \frac{\hbar c}{2e^2} = \frac{1}{2\alpha} e = \frac{137}{2} e = 68,5\ e,\;\; g = ng_0. \tag{3.27}$$

In conclusion, it is useful to emphasize that the functions [Eq. (3.24)] are defined for all the values of the Euler angles (namely, $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq 2\pi$, $0 \leq \chi \leq 2\pi$). These intervals are good for all angles, including θ, which is a so-called normal rotation angle, just as φ or χ is. Thus, the north pole is $\theta = 0$ and the south pole $\theta = 2\pi$. This fact seems shocking, but it is the reason for which the interval $0 \leq \theta \leq \pi$ is generally introduced, in order to obtain the univocity of Euler angles. But actually, it is better to describe the rotation group not in the physical space $\mathbb{R}^3$, but in $\mathbb{R}^4$, which is the SU^2 space and the space of the Euler-Olinde-Rodrigues parameters (Cartan, 1938; Lochak, 1959):

$$\begin{aligned} \xi_1 &= \sin\frac{\theta}{2}\cos\frac{\varphi-\chi}{2}, \quad \xi_2 = \sin\frac{\theta}{2}\sin\frac{\varphi-\chi}{2} \\ \xi_3 &= \cos\frac{\theta}{2}\sin\frac{\varphi+\chi}{2}, \quad \xi_4 = \cos\frac{\theta}{2}\cos\frac{\varphi+\chi}{2}. \end{aligned} \tag{3.28}$$

All is uniform in $\mathbb{R}^4$, including these parameters, the group representations, and the Euler angles. Now we must introduce the monopole harmonics with spin, obtained by the Clebsch-Gordan procedure (Lochak, 1985a,b, 1995):

$$\begin{aligned} \Omega_j^{m',m}(+) = \Omega_j^+ &= \begin{pmatrix} \left(\dfrac{j+m}{2j+1}\right)^{1/2} Z_j^{m',m-1} \\[2ex] \left(\dfrac{j-m+1}{2j+1}\right)^{1/2} Z_j^{m',m} \end{pmatrix}, \\ \Omega_j^{m',m}(-) = \Omega_j^- &= \begin{pmatrix} \left(\dfrac{j-m+1}{2j+1}\right)^{1/2} Z_j^{m',m-1} \\[2ex] -\left(\dfrac{j+m}{2j+1}\right)^{1/2} Z_j^{m',m} \end{pmatrix}. \end{aligned} \tag{3.29}$$

These harmonics correspond to the eigenvalues $k = j \pm 1/2$ of the total angular momentum $\mathbf{J}$. In the following discussion, we shall use the abbreviation Ω_j^-, Ω_j^-, as well as several relations, the first of which is directly deduced from Eq. (3.29):

$$\mathbf{J}^2\Omega_{j-1}^+ = \hbar^2 k(k+1)\,\Omega_{\mathbf{j}-1}^+,\ \mathbf{J}^2\Omega_j^- = \hbar^2 k(k+1)\,\Omega_j^-. \tag{3.30}$$

The others are deduced from recurrence relations between generalized spherical functions (Gelfand et al., 1963):

$$\begin{aligned} \mathbf{s}.\,\widehat{\mathbf{r}}\,\Omega^{+}_{j-1} &= \cos\Theta'\,\Omega^{+}_{j-1} + \sin\Theta'\,\Omega^{-}_{j} \\ \mathbf{s}.\,\widehat{\mathbf{r}}\,\Omega^{-}_{j} &= \sin\Theta'\,\Omega^{+}_{j-1} - \cos\Theta'\,\Omega^{+}_{j} \\ \cos\Theta' &= \frac{m'}{j} = \frac{D}{j},\quad \widehat{\mathbf{r}} = \frac{\mathbf{r}}{r}. \end{aligned} \tag{3.31}$$

The angle Θ' is the vertex half-angle of the Poincaré-cone (previously shown in Figure 1.2) because $\hbar m'$ is the projection of the total orbital momentum $\hbar j$ on the symmetry axis of the system, as defined by the monopole and the Coulombian center. We already knew that in the classical case (as discussed in Chapter 1), and we shall find it again at the geometrical limit of quantum theory.

3.5 THE INTERACTION BETWEEN A MONOPOLE AND AN ELECTRIC COULOMBIAN POLE (RADIAL FUNCTIONS)

The calculation of radial functions is based on the wave equations [Eq. (3.4)]. We consider the ξ-equation [Eq. (3.4)] with $W = 0$, making $B = 1/e\ \mathbf{B}$, where $\mathbf{B}$ is given by Eqs. (1.26) and (1.29), and looking for a solution with an angular momentum $k = j - 1/2$, taking into account Eq. (3.25). The ξ-equation becomes

$$\frac{i}{c}\frac{\partial\xi}{\partial t} = \mathbf{s}.(i\nabla - m'\mathbf{B})\xi. \tag{3.32}$$

To apply a classical integration method of the hydrogen atom in Dirac's theory, we introduce in Eq. (3.30) the following expansion for ξ, where $F^{\pm}(r)$ are the radial functions that we want:

$$\xi = e^{-i\omega t}\left[F^{+}_{j-1}(r)\Omega^{+}_{j-1} + F^{-}_{j}(r)\Omega^{-}_{j}\right]. \tag{3.33}$$

We find, multiplying by $\mathbf{s}.\,\widehat{\mathbf{r}}$,

$$\frac{\omega}{c}(\mathbf{s}.\,\widehat{\mathbf{r}})\left(F^{+}_{j-1}\Omega^{+}_{j-1} + F^{-}_{j}\Omega^{-}_{j}\right) = (\mathbf{s}.\,\widehat{\mathbf{r}})\ \mathbf{s}.(i\nabla - m'B)\left(F^{+}_{j-1}\Omega^{+}_{j-1} + F^{-}_{j}\Omega^{-}_{j}\right). \tag{3.34}$$

Using the equalities [Eqs. (1.26) and (3.18)], and the algebraic relation, we get

$$(\mathbf{s}.\mathbf{A}_1)(\mathbf{s}.\mathbf{A}_2) = \mathbf{A}_1.\mathbf{A}_2 + i(\mathbf{A}_1 \times \mathbf{A}_2).\,\mathbf{s} \tag{3.35}$$

Consequently, Eq. (3.34) takes the form

$$\frac{dF^{+}_{j-1}}{dr}\Omega^{+}_{j-1} + \frac{dF^{-}_{j}}{dr}\Omega^{-}_{j} = \left[\frac{1}{r}(\Lambda.\mathbf{s}) - \left(\frac{m'}{r} + i\frac{\omega}{c}\right)\ (\mathbf{s}.\,\mathbf{r})\right]\left(F^{+}_{j-1}\Omega^{+}_{j-1} + F^{-}_{j}\Omega^{-}_{j}\right). \tag{3.36}$$

We know, from Eqs. (3.18) and (3.20), that

$$\Lambda^2\Omega^{\pm} = \hbar^2 j(j+1)\Omega^{\pm},\ \ \mathbf{J}^2\Omega^{\pm} = \hbar^2 k(k+1)\Omega^{\pm}(k = j - 1/2), \tag{3.37}$$

and

$$(\Lambda.\,\mathbf{s})\Omega^{+}_{j-1} = (j-1)\Omega^{+}_{j-1}\,,\ \ (\Lambda.\,\mathbf{s})\Omega^{-}_{j} = -(j+1)\Omega^{-}_{j}\,. \tag{3.38}$$

Multiplying Eq. (3.36) on the left by Ω^{+}_{j-1} and Ω^{+}_{j-1} in succession, and integrating on the angles, we can eliminate $\Omega^{\pm}$. Using Eq. (3.31), we find

$$\begin{aligned}&\left[\frac{d}{dr} + \frac{1}{r} - \frac{j}{r}s_3 + \left(\frac{m'}{r} + i\frac{\omega}{c}\right)e^{-is_2(\Theta'/2)}s_3 e^{+is_2(\Theta'/2)}\right]F = 0,\\ &F(r) = \begin{pmatrix} F^{+}_{j-1}(r) \\ F^{-}_{j}(r)\end{pmatrix}.\end{aligned} \tag{3.39}$$

At this point, let us introduce functions $B^{+}_{j-1}(r)$, $B^{-}_{j}(r)$ such that

$$F = \frac{e^{is_2(\pi/4-\Theta'/2)}}{r}B,\ \ B = \begin{pmatrix} B^{+}_{j-1}(r) \\ B^{-}_{j}(r)\end{pmatrix}. \tag{3.40}$$

Eq. (3.39) now becomes

$$\left(\frac{d}{dr} - \frac{l}{r}s_3 + i\frac{\omega}{c}s_1\right)B = 0,\ \ l = j\sin\Theta' = \sqrt{j^2 - m'}. \tag{3.41}$$

We see that l is the projection of the total orbital angular momentum (monopole + field) on the plane orthogonal to the axis of the Poincaré cone (Poincaré, 1896). Differentiating Eq. (3.41), we obtain the Bessel equations (Ince, 1956):

$$\frac{d^2B^{+}_{j-1}}{dr^2} + \left[\left(\frac{\omega}{c}\right)^2 - \frac{l(l-1)}{r^2}\right]B^{+}_{j-1} = 0,\ \ \frac{d^2B^{-}_{j}}{dr^2} + \left[\left(\frac{\omega}{c}\right)^2 - \frac{l(l+1)}{r^2}\right]B^{-}_{j} = 0. \tag{3.42}$$

Using Eq. (3.41) and the recurrence formula, we get

$$zJ'_{\lambda}(z) + \lambda J_{\lambda}(z) = zJ_{\lambda-1}(z). \tag{3.43}$$

Finally, we have

$$B = \left(r\frac{\omega}{c}\right)^{1/2} \begin{pmatrix} iJ_{l-1/2}\left(r\frac{\omega}{c}\right) \\ J_{l+1/2}\left(r\frac{\omega}{c}\right) \end{pmatrix}. \tag{3.44}$$

Inserting this result in Eq. (3.40) and then in Eq. (3.33), we obtain the ξ spinor. A similar calculation would give the η spinor.

3.6 SOME GENERAL REMARKS

Here are some general remarks about this discussion so far:

1. Eq. (3.4) gives the correct expressions [i.e., Eq. (3.15)] for the angular momentum of a monopole in a coulombian field.
2. The Dirac relation for the product of an electric and a magnetic charge is deduced from our equation in a more precise form [i.e., Eqs. (3.26) and (3.27)], and the radial functions are also deduced from the equation. They are the same as those found for an electric charge in the field of an infinitely heavy monopole (Kazama, Yang, & Golhaber, 1977).
3. The classical analogy will be explained further in another chapter of this book.
4. ω is not quantized: the monopole in a coulombian electric field is always in a ionizing state.This fact, predicted by Dirac, might be *a priori* guessed for two reasons: (1) It is suggested by the spiraling motion on the cone described in the classical case by Poincaré, and we know that our equation has the Poincaré equation as a classical limit. (2) The potential **B** given in Eq. (1.26) has an infinite string and as a result, the wave equation cannot have square integrable solutions.
5. The fundamental difference between other theories and ours lies in the fact that the present theory is the only one based on a pseudoscalar charge operator $C = g\gamma_5$ and in which the charge constant g is a scalar, because the pseudoscalar character is confined in the operator γ_5. This entails that g is separately P, T, C invariant. To test what this difference means, let us introduce a pseudoscalar constant g instead of the operator $C = g\gamma_5$, in Eq. (2.15), which becomes $\left(\partial_\mu - \frac{g}{\hbar c}B_\mu\right)\Psi = 0$ (which is without i

because G_μ is a pseudovector). From (Lochak 5), Eq. (3.4) becomes $\left[\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{s}.\nabla - i\frac{g}{\hbar c}(W + \mathbf{s}.\mathbf{B})\right]\xi = 0$, $\left[\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{s}.\nabla - i\frac{g}{\hbar c}(W - \mathbf{s}.\mathbf{B})\right]\eta = 0$ with a difference with respect to Eq. (3.4). Both equations now have the same sign before i. This difference seems small, but actually it is important because, whereas the ξ and η equations exchange between themselves under the P and T transformations, as in the above mentioned system [Eq. (3.4)], the charge conjugation is now $C: g \to -g$, $-is_2\xi^* \to \eta$, $is_2\eta^* \to \xi$. The monopole and the antimonopole are thus not only chiral conjugated, they have opposite charges. Therefore, they can constitute pairs of magnetic charges and, by their masslessness, their annihilation induces a giant polarization. These particules are not true monopoles; rather, they are massless electric particles, "disguised in magnetic monopoles" (as stated in the Foreword and Lochak, 1985).

3.7 THE GEOMETRICAL OPTICS APPROXIMATION. BACK TO THE POINCARÉ EQUATION

Now we must verify that we have found the correct Poincaré equation and the Birkeland effect. Let us introduce in Eq. (3.4) the following expression of the spinor ξ:

$$\xi = a\, e^{iS/\hbar}, \tag{3.45}$$

where a is a two-component spinor and S a phase. At zero order in $\hbar$, we have

$$\left[\frac{1}{c}\left(\frac{\partial S}{\partial t} - gW\right) - \left(\nabla S + \frac{g}{c}\mathbf{B}\right).\,\mathbf{s}\right] a = 0, \tag{3.46}$$

which is a homogeneous system with respect to a. A necessary condition for a nontrivial solution is

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} - gW\right)^2 - \left(\nabla S + \frac{g}{c}\mathbf{B}\right)^2 = 0, \tag{3.47}$$

which is a relativistic Jacobi equation with zero mass, and we can define the kinetic energy, the impulse, and the linear Lagrange momentum as follows:

$$E = -\frac{\partial S}{\partial t} + gW, \quad \mathbf{p} = \nabla S + \frac{g}{c}\mathbf{B}, \quad \mathbf{P} = \nabla S. \tag{3.48}$$

The Hamiltonian function will equal

$$H = c\sqrt{\left(\mathbf{P} + \frac{g}{c}\mathbf{B}\right)^2} - gW, \tag{3.49}$$

and a classical calculation gives as an equation of motion:

$$\frac{d\mathbf{p}}{dt} = g\left(\nabla W + \frac{\partial \mathbf{B}}{\partial t}\right) - \frac{g}{c}\mathbf{v} \times curl\,\mathbf{B}, \tag{3.50}$$

which gives the classical form

$$\frac{d\mathbf{p}}{dt} = g\left(\mathbf{H} - \frac{1}{c}\mathbf{v} \times \mathbf{E}\right). \tag{3.51}$$

But we must not forget that the mass of our particle equals zero, so $\mathbf{v}$ is the velocity of light and we cannot write $\mathbf{p} = m\mathbf{v}$. But the equality $\mathbf{p} = (E/c^2)\,\mathbf{v}$ still holds when the energy E is a constant, which will be the case in a coulombian electric field. So we have

$$\frac{d^2\mathbf{p}}{dt^2} = -\lambda\frac{1}{r^3}\frac{1}{c}\frac{d\mathbf{r}}{dt} \times \mathbf{r}, \quad \lambda = \frac{egc}{E} \tag{3.52}$$

This is exactly the Poincaré equation [Eq. (1.2), given previously in Chapter 1], with a minus sign because we chose the left monopole. Now, starting from the right monopole [i.e., from the second equation in Eq. (3.4)] and with the same approximation:

$$\eta = b\, e^{iS/\hbar}, \tag{3.53}$$

we find the following equation for b:

$$\left[\frac{1}{c}\left(\frac{\partial S}{\partial t} + gW\right) - \left(\nabla S - \frac{g}{c}\mathbf{B}\right).\,\mathbf{s}\right] b = 0. \tag{3.54}$$

Of course, Eq. (3.54) gives the same Poincaré equation [Eq. (3.52)] with a plus sign before λ.

3.8 THE PROBLEM OF THE LINK BETWEEN A LEPTONIC MAGNETIC MONOPOLE, A NEUTRINO, AND WEAK INTERACTIONS

The problem being explored in this chapter may be summarized as follows:

1. The Weyl representation splits the massless Dirac equation into two independent, two-component equations, which are considered since 1956 (by Lee and Yang and Landau) as the "neutrino two-component theory": one equation describes the neutrino and the other one the anti-neutrino.
2. We have shown that the massless Dirac equation admits a second gauge invariance: the *chiral gauge*. In addition, we have shown that not only C-symmetry, but also P-symmetry has the important property of being able to exchange between themselves the two Weyl equations, left and right, respectively, which implies the chirality of the neutrino and the anti-neutrino[8].
3. Now, we have proved that *the chiral gauge invariance of the massless Dirac equation entails a new electromagnetic interaction which corresponds to a magnetic monopole*. And obviously there is no other possibility. The new pair of Weyl-like equations with these new interaction terms remain separated as with a free field. And it must be emphasized that our massless monopole is a consequence of the second gauge invariance of the Dirac equation. It is profoundly rooted in electronics and electromagnetism, and for this reason, it is very different from the monopoles with enormous masses predicted by other theories. And the principal difference is that our monopole leaves observed characteristic tracks, it is created in several laboratories, and it gives physical observable consequences.
4. Now, the neutrino appears in this theory as a magnetic monopole with a zero charge. Remember that this charge is equal to n times a unit charge (including $n = 0$). And the laws of symmetry are identical. For this reason, we have presented the hypothesis that these leptonic monopoles, thanks to the neutrino symmetry, manifest low-energy interactions. This hypothesis was largely developed by Harald Stumpf, and we put forward here a couple of experimental arguments:

[8] There is a curious anecdote concerning this property. When Hermann Weyl, at the end of the 1920s, found his representation of the Dirac equation, he noticed, *in the massless case*, the splitting into the two equations that we are discussing (the difference is that we introduce the interaction with an electromagnetic field). So, he said, each half of the split equations acquires an independent sense. Pauli objected to this because such an equation is not P-invariant. That is true, of course, but he neglected the fact that there were, actually, *two equations*, which together were P-invariant, and it was unknown at that time that they were left and right. The funny part of this story is that Pauli a little later predicted the existence of the neutrino—i.e., the clue to the problem that was finally untangled a quarter of a century later. The sad part of the story is, that if Pierre Curie—the discoverer, if not the solver, of these problems—had been alive, perhaps all would have been evident to him from the very beginning.

a. A beta radioactive sample (normally emitting neutrinos), submitted to a magnetic field, emits leptonic monopoles (Ivoilov, 2006).
b. The lifetime of a beta radioactive sample is reduced when it is irradiated by leptonic monopoles (Ivoilov, 2006).
c. A great quantity of neutrinos is emitted by the sun (because of the great number of low-energy reactions). I have suggested the hypothesis that some of them could be excited as leptonic monopoles by strong solar magnetic fields. If this is the case, most of these monopoles would have to be trapped on the sun by the same magnetic fields, which could be a new hypothesis that could explain the lack of solar neutrinos received by the Earth. Nevertheless, some of these monopoles could escape and then follow trajectories directed toward the Earth. In such a case, they must follow the lines of the magnetic field directed to the Earth's magnetic poles. So, when the explorer Jean-Louis Etienne embarked on an expedition to the North Pole, we gave him some X-ray films, currently used in laboratories to register leptonic monopoles. And we have found on these films exactly the same characteristic lines of monopoles (Bardout et al., 2007).

3.9 SOME QUESTIONS ABOUT THE DIRAC FORMULA AND OUR FORMULA

At this point, let us recall this equation:

$$D = \frac{eg}{\hbar c} = \frac{n}{2}, \ (1.19), \ \text{and}: \ D = \frac{eg}{\hbar c} = m' = -j, -j+1, \ldots, j-1, j; \ \left(j = \frac{n}{2}\right), \tag{3.26}$$

Dirac's conclusions and ours are different. Dirac was looking for the reason why all the electric charges that appear in the physical world are equal to either the electron charge or to a mutiple, and he was happy to find that, by virtue of his formula, an arbitrary electric charge e must equal $e = n\frac{\hbar c}{2g}$. Therefore, if there is even only one monopole in the world, all the elecric charges will be multiples of a unit charge that depends on the charge of this monopole.

Our position is different. We have a theory concerning a magnetic monopole, and we ask the question: What happens if that monopole interacts with an electric charge? We are, in principle, able to answer this question because we have a wave equation [namely, Eq. (2.15)]. But *the answer depends on the value of these charges*, contrary to what happens with two electric charges, which can always interact without any condition: the reason is that in our case, one charge is a scalar and the other is a pseudoscalar, which was shown in the Dirac case because Eqs. (1.12) and (1.25) present the same difficulty.

Thus, we find that Eq. (3.26) has an evident affinity with the Dirac condition, with the difference that in our case, the *electric charge—not the magnetic one—is given*, so that it seems that we must write Eq. (3.26) in inverse order (with α being the fine structure constant):

$$\begin{aligned} & g = m'\frac{\hbar c}{e} = em'\frac{\hbar c}{e^2} = e\frac{m'}{\alpha} = 137 \;\; em' \\ & \text{with}: \quad m' = -j, -j+1, \ldots, j-1, j \quad \left(j = \frac{n}{2}\right). \end{aligned} \tag{3.54}$$

Therefore, if we consider a magnetic charge g striking a particle with an electric charge e, the collision will be possible only when the charge g of the magnetic particle obeys the condition [Eq. (3.54)], depending on the electric charge and on the angular momentum (more precisely, on its *projection* on the symmetry axis): so that not only the charge but even the momentum must be good. And the problem is still more complicated because there are electric particles with greater charges: for instance, atomic nuclei with charges Ne, so that Eq. (3.54) becomes

$$g = m'\frac{\hbar c}{Ne} = em'\frac{\hbar c}{Ne^2} = e\frac{m'}{N\alpha} = e\frac{137}{N}m'. \tag{3.55}$$

So, we must conclude that, on account of the relation [Eq. (3.26)], it is impossible to conclude that the electric and the magnetic charges are both conservative quantities because the conservation laws deduced from their respective wave equations are not verified in the collisions.

But we have strong theoretical and experimental arguments in favor of an absolute conservation of the electric charge, at least in the frame of the actually recognized electromagnetic laws. Thus, it seems that we are obliged to admit that, despite the fact that Eqs. (2.16) and (3.4) of the leptonic magnetic monopoles seem to be correct for symmetry laws, for the link to weak interactions and for electromagnetic interactions with continuous fields, something is missing in the description of the interaction between magnetic and electric charges.

It is highly improbable that this problem results from a defect in Eq. (2.16) or (3.4) because the preceding arguments could be developed in the Dirac case, starting from Eq. (1.19), as in ours, starting from Eq. (3.26). And these relations are mutually reinforced not only by their analogy, but also because they are confirmed by different arguments.

It seems evident that the difficulties are in the facts, not in the method. Manifestly, these equations need to be generalized by the presence of operators that can describe quantum transitions between the different states, defined by the preceding conditions, which is not presently the case.

CHAPTER 4

Nonlinear Equations. Torsion and Magnetism

Until now, we have seen only linear equations of a magnetic monopole: Eqs. (2.16) and (3.4). This is quite natural, because our theory concerns the magnetic slope of the Dirac theory of the electron, which is itself linear. Actually, the strangeness of this theory is not the linearity, which is normal in quantum mechanics, but rather the fact that the monopole so described is *massless* for algebraic reasons, which plays a basic role in the theory and cannot be easily dismissed. It must be emphasized that I personally profoundly dislike it because of the strangeness of the fact in itself, and, I must confess, because I am a member of the de Broglie school, which always hated masslessness, even applied to the photon, from which this peculiarity was eliminated.

Nevertheless, at first glance it seems difficult to avoid this peculiarity in the case of our magnetic monopole because it is a consequence of the *chiral gauge invariance*, which itself lies at the origin of all the results of the theory, as follows:

- The conservation of magnetism and the correct electromagnetic interaction of a monopole
- The classical limit, which gives the Poincaré equation and the analogy with a symmetric top
- The accordance with the symmetry laws predicted by Pierre Curie (1894a,b), which are experimentally verified (first of all, the chiral symmetry)
- A more precise form of the Dirac relation between electric and magnetic charges
- The analogy between neutrinos and leptonic monopoles, the latter being considered as magnetically excited neutrinos and obeying the same laws of symmetry
- The influence of monopoles on the lifetime of β radioactivity

Thus, we have reason to believe that the γ_5 gauge is unavoidable. Therefore, if we wish to define a mass term, we must look for a new way to do so without abandoning chiral invariance. We have found such a way: namely, *nonlinearity,* because we have already found a *nonlinear chiral invariant,* which was given as Eq. (2.20) in Chapter 2 and Eq. (3.10) in Chapter 3 of this book.

As a result, we can introduce in the Lagrangian a function $F(\rho)$ of the chiral invariant as a mass term. Of course, we could just as easily introduce a function on the norm of the electric or magnetic currents: $J_\mu J_\mu$ or $\Sigma_\mu \Sigma_\mu$ (as cited in Chapter 2), as Heisenberg did in his nonlinear theory (Heisenberg, 1953, 1954, 1966; Dürr *et al.*, 1959; Borne, Lochak, & Stumpf, 2001; Lochak., 1985), but we know that these norms are, including the sign, equal to ρ^2 [Eq. (2.24)].

4.1 A NONLINEAR MASSIVE MONOPOLE

First, let us write the following Lagrangian (Lochak, 1985) in the Dirac representation[9]:

$$L = \overline{\psi}\gamma_\mu[\partial_\mu]\psi - \frac{g}{\hbar c}\overline{\psi}\gamma_\mu\gamma_5 B_\mu\psi + i\frac{\mathrm{M}(\rho)c}{\hbar}, \tag{4.1}$$

where ρ is given by Eq. (2.20) and $\mathrm{M}(\rho)$ is a scalar function of ρ with the dimension of a mass.

The corresponding equation is

$$\gamma_\mu\left(\partial_\mu - \frac{g}{\hbar c}\gamma_5 B_\mu\right)\Psi + i\frac{\mu(\rho)c}{\hbar}\frac{\omega_1 - i\gamma_5\omega_2}{\sqrt{\omega_1^2+\omega_2^2}}\Psi = 0, \quad \mu(\rho) = \frac{d\mathrm{M}(\rho)}{d\rho}. \tag{4.2}$$

In the Weyl representation, we get

$$\begin{aligned} L = {} & \xi^+\left(\frac{1}{c}[\partial_t] - \frac{g}{\hbar c}W\right)\xi - \xi^+\mathbf{s}.\left([\nabla] + \frac{g}{\hbar c}\mathbf{B}\right)\xi + \\ & +\eta^+\left(\frac{1}{c}[\partial_t] + \frac{g}{\hbar c}W\right)\eta + \eta^+\mathbf{s}.\left([\nabla] - \frac{g}{\hbar c}\mathbf{B}\right)\eta + i\frac{\mathrm{M}(\rho)c}{\hbar}, \end{aligned} \tag{4.3}$$

[9] Here, we use the Costa de Beauregard convention $[\partial] = \overrightarrow{\partial} - \overleftarrow{\partial}$.

which gives the following equations:

$$\begin{aligned}&\frac{1}{c}\partial_t\xi - \mathbf{s}.\nabla\xi - i\frac{g}{\hbar c}(W+\mathbf{s}.\,\mathbf{B})\xi + i\frac{\mu(\rho)c}{\hbar}\sqrt{\frac{\eta^+\xi}{\xi^+\eta}}\,\eta = 0\\ &\frac{1}{c}\partial_t\eta + \mathbf{s}.\nabla\eta + i\frac{g}{\hbar c}(W-\mathbf{s}.\,\mathbf{B})\eta + i\frac{\mu(\rho)c}{\hbar}\sqrt{\frac{\xi^+\eta}{\eta^+\xi}}\,\xi = 0\end{aligned} \qquad \left(\mu(\rho) = \frac{d\mathrm{M}(\rho)}{d\rho}\right) \tag{4.4}$$

These equations are chiral-invariant, like the linear equation. The magnetic current [Eq. (2.23)] is conserved, and, owing to Eq. (2.24), the equations are *PTC*-invariant (Lochak, 1997a, b) and the isotropic chiral currents [Eq. (3.8)] are separately conserved. In spite of that, Eq. (4.4) is generally coupled, as opposed to Eq. (3.4). But this coupling is not strong: If the degree of $\mu(\rho)$ is greater than 1, the nonlinear term vanishes when $\rho = 2\sqrt{(\xi^+\eta)(\eta^+\xi)} = 0$, which happens either for $\xi = 0$ or $\eta = 0$, or on the light cone (the Majorana case), which will be examined later in this chapter.

Now one can see that, in Eq. (4.4), ξ and η are phase-independent. This is why we can consider plane waves with different frequencies ω and ω', as well as wave numbers k and k', for ξ and η:

$$\xi = \mathrm{a}\ \mathrm{e}^{\mathrm{i}(\omega\ t-\mathbf{k}.\mathbf{r})};\ \eta = b\ \mathrm{e}^{\mathrm{i}(\omega' t-\mathbf{k}'.\mathbf{r})}. \tag{4.5}$$

Introducing these expressions in Eq. (4.4) *without external field*, we find (Lochak, 1985, 1995a,b)

$$\begin{aligned}&\left(\frac{\omega}{\mathrm{c}} + \mathbf{s}.\mathbf{k}\right)a + \frac{\mu(\rho)c}{\hbar}\sqrt{\frac{b^+a}{a^+b}}\,b = 0\\ &\left(\frac{\omega'}{\mathrm{c}} - \mathbf{s}.\mathbf{k}'\right)b + \frac{\mu(\rho)c}{\hbar}\sqrt{\frac{a^+b}{b^+a}}\,a = 0.\end{aligned} \tag{4.6}$$

If we multiply the first equation by $\left(\frac{\omega'}{\mathrm{c}} - \mathbf{s}.\mathbf{k}'\right)$, with the following definitions:

$$\begin{aligned}&\left(\frac{\omega'}{\mathrm{c}} - \mathbf{s}.\mathbf{k}'\right)\ \left(\frac{\omega}{\mathrm{c}} + \mathbf{s}.\mathbf{k}\right) = \Omega + \mathbf{s}.\mathbf{K};\ \ \Omega = \frac{\omega\omega'}{c^2} - \mathbf{k}.\mathbf{k}',\\ &\mathbf{K} = \frac{1}{c}(\omega'\mathbf{k} - \omega\mathbf{k}') + i\mathbf{k}\times\mathbf{k}',\end{aligned} \tag{4.7}$$

we have

$$(\boldsymbol{\Omega} + \mathbf{s}.\mathbf{K})a + \frac{\mu(\rho)c}{\hbar}\sqrt{\frac{b^{+}a}{a^{+}b}}\left(\frac{\omega'}{c} - \mathbf{s}.\mathbf{k}'\right)b = 0. \tag{4.8}$$

Then, owing to Eq. (4.6), we find

$$\left[\boldsymbol{\Omega} + \mathbf{s}.\mathbf{K} - \left(\frac{\mu(\rho)c}{\hbar}\right)^{2}\right]a = 0, \tag{4.9}$$

and finally, we must make the determinant of this equation zero to find a nontrivial solution, which gives the dispersion relation

$$\left(\boldsymbol{\Omega} - \left(\frac{\mu(\rho)c}{\hbar}\right)^{2}\right)^{2} - \mathbf{K}^{2} = 0. \tag{4.10}$$

We shall find a more explicit expression going back to Eq. (4.7), from which

$$\mathbf{K}^{2} = \frac{1}{c^{2}}(\omega'\mathbf{k} - \omega\mathbf{k}')^{2} - (\mathbf{k} \times \mathbf{k}')^{2}, \tag{4.11}$$

and hence it is easy to deduce the following:

$$\boldsymbol{\Omega}^{2} - \mathbf{K}^{2} = \left(\frac{\omega^{2}}{c^{2}} - k^{2}\right)\left(\frac{\omega'^{2}}{c'^{2}} - k'^{2}\right). \tag{4.12}$$

Thus, we get the dispersion relation

$$\left(\frac{\omega^{2}}{c^{2}} - \mathbf{k}^{2}\right)\left(\frac{\omega'^{2}}{c^{2}} - \mathbf{k}'^{2}\right) - 2\left(\frac{\omega\ \omega'}{c^{2}} - \mathbf{k}.\mathbf{k}'\right)\left(\frac{\mu(\rho)c}{\hbar}\right)^{2} + \left(\frac{\mu(\rho)c}{\hbar}\right)^{4} = 0\,. \tag{4.13}$$

Now, let us take the case of a homogeneous equation in ξ and η:

$$\mathrm{M}(\rho) = \mu_0\,\rho; \quad \mu(\rho) = \mu_0 = Const. \tag{4.14}$$

Owing to Eq. (4.13), we easily find two interesting kinds of waves:

1. $\omega = \omega'$, $\mathbf{k} = \mathbf{k}'$: Both monopoles have the same phase, and the dispersion relation reduces to

$$\frac{\omega^{2}}{c^{2}} = k^{2} + \mu_0^{2}\ \left(k = \sqrt{\mathbf{k}^{2}}\right). \tag{4.15}$$

This is the ordinary dispersion relation of a massive particle, known as a *bradyon*.

2. On the other hand, if we have $\omega = -\omega'$, $\mathbf{k} = -\mathbf{k}'$, the phases have opposite signs, and the dispersion relation becomes

$$\frac{\omega^2}{c^2} = k^2 - \mu_0^2 \tag{4.16}$$

This is the dispersion relation of a supraluminal particle, known as a *tachyon*.

The wave equations [Eqs. (4.2) and (4.4)] seem to be the first in which tachyons appear without any ad hoc condition, but only as a particular solution among others. These nonlinear equations can be considered in different ways, which were described in Lochak (2003). Let us state once more that *the chiral components of the nonlinear equations [Eq. (4.2)] of a monopole in a coulombian electric field cannot be separated,* as they were in the linear case [Eq. (3.4)].

4.2 THE NONLINEAR MONOPOLE IN A COULOMBIAN ELECTRICAL FIELD

We shall see shortly that, in a coulombian electrical field with the pseudopotential [Eq. (1.26)], not only the linear equations [Eq. (3.4)], but also the nonlinear equations [Eq. (4.4)] admit the same angular operator [Eq. (3.14)], as an integral of motion.

For technical reasons, we shall collect the operators [Eq. (3.15)] into a unique operator in the Dirac representation, and we shall introduce the following classical vectorial notation:

$$\mathbf{J} = \hbar\left[\mathbf{r} \times (-i\nabla + \gamma_4 D\mathbf{B}) + \gamma_4 D\frac{\mathbf{r}}{r} + \frac{1}{2}\mathbf{S}\right]; \quad \mathbf{S} = \begin{pmatrix} \mathbf{s} & 0 \\ 0 & \mathbf{s} \end{pmatrix}; \qquad D = \frac{eg}{\hbar c}; \ \mathbf{B} \to eB. \tag{4.17}$$

Of course, the commutation rules [Eq. (3.17)] are satisfied by the components of Eq. (4.17), and we shall prove that $\mathbf{J}$ is an integral of motion, but we must be more careful than in the linear case described in Chapter 3 because of the presence of a nonlinear term in the Hamiltonian. So we return to the definition of a first integral, which is not a commutation rule but rather the definition: *the mean value of the operator* $\mathbf{J}$ *is a constant in* *virtue of the wave equations [Eq. (4.2) or (4.4)]*. To do so, we introduce the Dirac form of quantum equations in a vectorial formulation:

$$\frac{1}{c}\frac{\partial\psi}{\partial t} = H\psi; \quad H = \alpha.\nabla + i\,D\,\mathbf{S}.\,\mathbf{B} + i\,\mu(\rho)\;(\omega_1\alpha_4 + \omega_2\alpha_5) \tag{4.18}$$

$$\alpha = \begin{pmatrix} 0 & \mathbf{s} \\ \mathbf{s} & 0 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \gamma_4; \quad \mathbf{S} = \begin{pmatrix} \mathbf{s} & 0 \\ 0 & \mathbf{s} \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \gamma_5$$

$$\sigma = \sigma_4\alpha, \quad -i\alpha_1\alpha_2\alpha_3 = \sigma_4, \quad \alpha_1\alpha_2\alpha_3\alpha_4 = \alpha_5, \quad \mathbf{s} \;=\; \text{Pauli matrices.} \tag{4.19}$$

Now we must prove that

$$\frac{d}{dt}\int \Psi^{+}\mathbf{J}\,\Psi\;dxdydz = \int\left(\frac{\partial\Psi^{+}}{\partial t}\mathbf{J}\,\Psi + \Psi^{+}\mathbf{J}\,\frac{\partial\Psi}{\partial t}\right)\;dxdydz$$
$$= i\int \Psi^{+}[H\mathbf{J} - \mathbf{J}H\,]\Psi =\; 0. \tag{4.20}$$

The classical $[H\,\mathbf{J} - \mathbf{J}\;H\,]$ commutator appears, but we must examine in more detail the following three terms:

$$\frac{1}{c}\frac{d}{dt}\int \psi^{+}\mathbf{J}\psi d\mathrm{v} = \mathbf{P} + \mathbf{Q} + \mathbf{R} \tag{4.21}$$

which correspond to the x, y, z components of $\mathbf{J}$. For instance, we have

$$\mathbf{P}_x = \int \psi^{+}(\mathbf{J}_x\,\boldsymbol{\alpha}.\nabla - \boldsymbol{\alpha}.\nabla\,\mathbf{J}_x)\psi d\mathrm{v}$$
$$\mathbf{Q}_x = iD\int \psi^{+}\left(\mathbf{J}_x\;\frac{z(\sigma_1 y - \sigma_2 x)}{r(x^2+y^2)} - \frac{z(\sigma_1 y - \sigma_2 x)}{r(x^2+y^2)}\,\mathbf{J}_x\right)\psi d\mathrm{v} \tag{4.22}$$
$$\mathbf{R}_x = \int \psi^{+}[\mathbf{J}_x\,(\omega_1\alpha_4 + \omega_2\alpha_5) - (\omega_1\alpha_4 + \omega_2\alpha_5)\,\mathbf{J}_x]\psi d\mathrm{v}.$$

Now, recall that in Chapter 1, we had

$$B_x = \frac{e}{r}\frac{yz}{x^2+y^2}, \quad B_y = \frac{e}{r}\frac{-xz}{x^2+y^2}, \quad B_z = 0, \quad r = \sqrt{x^2+y^2+z^2}, \tag{1.26}$$

so

$$yB_z - B_y z = \frac{xr}{x^2+y^2} - \frac{x}{r}; \quad zB_x - xB_z = \frac{yr}{x^2+y^2} - \frac{y}{r}; \quad xB_y - yB_y = -\frac{z}{r}, \tag{4.23}$$

and the operator [Eq. (4.17)] becomes

$$\hbar^{-1}\mathbf{J}_x = -i(\mathbf{r}\times\nabla)_x + D\frac{xr}{x^2+y^2}\sigma_4 + \frac{1}{2}\sigma_1$$
$$\hbar^{-1}\mathbf{J}_y = -i(\mathbf{r}\times\nabla)_y + D\frac{yr}{x^2+y^2}\sigma_4 + \frac{1}{2}\sigma_2 \tag{4.24}$$
$$\hbar^{-1}\mathbf{J}_z = -i(\mathbf{r}\times\nabla)_z + \frac{1}{2}\sigma_3.$$

So Eq. (4.22) is split into "triads," corresponding to $\mathbf{J}_x$, $\mathbf{J}_y$, $\mathbf{J}_z$. For instance, we have for x:

$$\mathbf{P}_x = \int \psi^+(\mathbf{J}_x\,\boldsymbol{\alpha}.\nabla - \boldsymbol{\alpha}.\nabla\,\mathbf{J}_x)\psi dv$$
$$\mathbf{Q}_x = iD\int \psi^+\left(\mathbf{J}_x\,\frac{z(\sigma_1 y-\sigma_2 x)}{r(x^2+y^2)} - \frac{z(\sigma_1 y-\sigma_2 x)}{r(x^2+y^2)}\,\mathbf{J}_x\right)\psi dv$$
$$\mathbf{R}_x = \int \psi^+[\mathbf{J}_x\mu(\rho)\,(\omega_1\alpha_4+\omega_2\alpha_5) - \mu(\rho)(\omega_1\alpha_4+\omega_2\alpha_5)\,\mathbf{J}_x]\psi dv. \tag{4.25}$$

Here, we shall consider only this x-case. Introducing Eq. (4.21), we have

$$\mathbf{P}_x = P_1 + P_2 + P_3;\quad \mathbf{Q}_x = Q_1 + Q_2 + Q_3;\quad \mathbf{R}_x = R_1 + R_2 + R_3. \tag{4.26}$$

If we gather Eq. (4.18) to Eq. (4.26), we find

$$P_1 = -i\int \psi^+[(\mathbf{r}\times\nabla)(\boldsymbol{\alpha}.\nabla) - (\mathbf{r}\times\nabla)(\boldsymbol{\alpha}.\nabla)]\psi dv$$
$$= \mathrm{i}\textstyle\int \psi^+\left(\alpha_2\partial_z - \alpha_3\partial_y\right)\psi dv$$
$$P_2 = -D\int \psi^+\left[(\boldsymbol{\alpha}.\nabla)\;\sigma_4\frac{xr}{x^2+y^2}\right]\psi dv \tag{4.27}$$
$$P_3 = -\mathrm{i}\int \psi^+\left(\alpha_2\partial_z - \alpha_3\partial_y\right)\psi dv = -P_1 \;\rightarrow\; P_1 + P_3 = 0.$$

We know that $\mathbf{P}_x = P_1 + P_2 + P_3$, and we find the following for $\mathbf{Q}_x$:

$$Q_1 = D\int \psi^+\left(y\partial_z - z\partial_y\right)\frac{z(\sigma_1 y-\sigma_2 x)}{r(x^2+y^2)}\psi dv$$
$$Q_2 = 0 \tag{4.28}$$
$$Q_3 = D\int \psi^+\frac{\sigma_3 xz}{r(x^2+y^2)}\psi dv \;\rightarrow\; Q_1 + P_2 = -Q_3.$$

Hence we see that $[H\mathbf{J} - \mathbf{J}H] = 0$ *for the first three linear terms of the Hamiltonian [Eq. (4.18)], which ensures* a fortiori *the conservation of* $\mathbf{J}$ *in the linear case presented in Chapter 3*.

Now, only the nonlinear part remains, which reduces Eq. (4.25) to the following condition:

$$\mathbf{R}_x = \int \psi^+ [\mathbf{J}_x \mu(\rho)\ (\omega_1\alpha_4 + \omega_2\alpha_5) - \mu(\rho)(\omega_1\alpha_4 + \omega_2\alpha_5)\, \mathbf{J}_x]\psi dv. \tag{4.29}$$

At first, we have

$$\begin{aligned} R_1 = &- i \int \psi^+ [(\mathbf{r} \times \nabla)\mu(\rho)\ (\omega_1\alpha_4 + \omega_2\alpha_5) \\ &- \mu(\rho)(\omega_1\alpha_4 + \omega_2\alpha_5)(\mathbf{r} \times \nabla)]\psi dv, \end{aligned} \tag{4.30}$$

which easily takes the following form:

$$R_1 = -i \int \psi^+ (\mathbf{r} \times \nabla) F(\rho) \psi dv\ =\ \mathrm{i} \int \psi^+ (\mathbf{r} \times \nabla) F\left(\omega_1^2 + \omega_2^2\right) \psi dv, \tag{4.31}$$

where $F(\omega_1^2 + \omega_2^2)$ is a function, the expression of which is not important because $(\omega_1^2 + \omega_2^2)$ does not depend on the angles, so that $R_1 = 0$. Finally, we find the following for the other components:

$$\begin{aligned} R_2 &= D \int \psi^+ \frac{\mathbf{r}}{x^2 + y^2} [\sigma_4\ (\omega_1\alpha_4 + \omega_2\alpha_5) - (\omega_1\alpha_4 + \omega_2\alpha_5)\sigma_4\ ]\psi dv\ =\ 0 \\ R_3 &= \int \psi^+ \mu(\rho)[\ \sigma(\omega_1\alpha_4 + \omega_2\alpha_5) - (\omega_1\alpha_4 + \omega_2\alpha_5)\ \sigma]\psi dv\ =\ 0 \end{aligned} \tag{4.32}$$

because σ commutes and σ_4 anticommutes with α_4 and α_5.

So, the nonlinear equations [Eq. (4.2) or (4.3)] define the same angular momentum [Eq. (3.15)] as the linear equations. Therefore, the angular part must be the same in both cases; the difference is only in the radial factor.

4.3 CHIRAL GAUGE AND TWISTED SPACE. TORSION AND MAGNETISM

Let us take the particular case of Eq. (4.2) when $B_\mu = 0$; $\kappa(\rho) = \lambda\,\rho,\ \lambda = const$:

$$\gamma_\mu \partial_\mu \Psi + i\lambda(\omega_1 - i\gamma_5\omega_2)\Psi = 0. \tag{4.33}$$

Equivalent equations were considered by many researchers (Finkelstein, Lelevier, & Ruderman, 1951; Heisenberg, 1954; Dürr *et al.*, 1959; Weyl, 1950; Rodichev, 1961). Of these, Rodichev (1961) was the one to consider a space with an *affine connection,* and we shall briefly summarize this problem as follows:

1. No metric is introduced, and the theory is formulated only in terms of *connection coefficients* Γ^i_{rk}. One can define contravariant and covariant vectors T^i and T_i, and *covariant derivatives*:

$$\nabla_\mu T^i = \partial_\mu T^i + \Gamma^i_{r\mu} T^r; \quad \nabla_\mu T_i = \partial_\mu T_i - \Gamma^r_{i\mu} T_r. \tag{4.34}$$

2. Two important tensors are defined here[10], *curvature* and *torsion:*

$$-R^\mu_{\nu\sigma\lambda} = \partial_\sigma \Gamma^\mu_{\nu\lambda} - \partial_\lambda \Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\rho\sigma}\Gamma^\rho_{\nu\lambda} - \Gamma^\mu_{\rho\lambda}\Gamma^\rho_{\nu\sigma} \text{ and } S^\lambda_{[\mu\nu]} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \tag{4.35}$$

3. A *parallel transport* along a curve $x(t)$ is defined by $\nabla_\xi T = \xi^k \nabla_k T = 0$, $(\xi = x(t))$. A *geodesic line* is generated by the parallel transport of its tangent. Apart from a Euclidian space, a *geodesic rectangle is* broken by a gap in two terms: the first, in dt^2, depends on torsion, while the second, of the order of $o(dt^3)$, depends on curvature.
4. In a *twisted space* $(S^\lambda_{[\mu\nu]} \neq 0)$, a geodesic loop is an arc of helicoid, with a "thread" of the *second order:* the order of an area. Something similar happens in a *spin fluid:* the angular momentum of a droplet is of higher order than the spin (Costa de Beauregard, 1983; Weyl, 1950). Now, Rodichev considers the case of *a flat, twisted space:* with torsion $(\Gamma^\lambda_{[\mu\nu]} = S^\lambda_{[\mu\nu]} \neq 0)$ but straight geodesics $(\Gamma^\lambda_{(\mu\nu)} = 0)$, and with the following connection and covariant spinor derivative:

$$\Gamma_{\lambda[\mu\nu]} = S_{\lambda\mu\nu} = \Phi_{[\lambda\mu\nu]}; \quad \nabla_\mu \Psi = \partial_\mu \Psi - \frac{i}{4}\Phi_{[\lambda\mu\nu]}\gamma_\mu\gamma_\nu \Psi \tag{4.36}$$

and the following Lagrangian density:

$$L = \frac{1}{2}\left\{\overline{\psi}\, \gamma_\mu \nabla_\mu \psi - \left(\nabla_\mu \overline{\psi}\right) \gamma_\mu \psi\right\}. \tag{4.37}$$

[10] When $R^i_{qkl} = S^\lambda_{[\mu\nu]} = 0$, the space is Euclidian.

Translating the last formula in our notation, it gives:

$$L = \frac{1}{2}\left\{\overline{\psi}\,\gamma_\mu\partial_\mu\psi - \left(\partial_\mu\overline{\psi}\right)\,\gamma_\mu\psi - \frac{i}{2}\Phi_{[\mu\nu\lambda]}\overline{\psi}\gamma_\mu\gamma_\lambda\psi\right\}. \tag{4.38}$$

Introducing the axial dual vector $\Phi_\mu = \frac{i}{3!}\varepsilon_{[\mu\nu\lambda\sigma]}\Phi_{[\nu\lambda\sigma]}$, the Lagrangian becomes

$$L = \frac{1}{2}\{\overline{\psi}\,\gamma_\mu\partial_\mu\psi - \left(\partial_\mu\overline{\psi}\right)\gamma_\mu\psi - \Phi_\mu\overline{\psi}\gamma_\mu\gamma_5\psi\}, \tag{4.39}$$

which gives the following equation:

$$\gamma_\mu\left(\partial_\mu - \frac{1}{2}\Phi_\mu\gamma_5\right)\psi = 0. \tag{4.40}$$

With $\Phi_\mu = \frac{2g}{\hbar c}B_\mu$, this is *our equation* [Eq. (2.16)]. Let us note that Rodichev did not introduce Φ_μ as an external field: it was only a geometrical property. But in our case, we can say that a monopole plunged into an electromagnetic field induces a torsion in the surrounding space.

Rodichev ignored the monopole. He did not aim at the linear equation [Eq. (2.16)], but rather at a nonlinear equation, through the following Einstein-like action integral without an external field:

$$S = \int (L - bR)\; d^4x, \tag{4.41}$$

where L is given by Eq. (4.39), $b = Const$, $R =$ total curvature, and

$$R = \Phi_{[\lambda\mu\nu]}\Phi_{[\lambda\mu\nu]} = -6\Phi_\mu\Phi_\mu. \tag{4.42}$$

Hence, Eq. (4.41) becomes

$$S = \int \frac{1}{2}\{\left[\overline{\psi}\,\gamma_\mu\partial_\mu\psi - \left(\partial_\mu\overline{\psi}\right)\gamma_\mu\psi - \Phi_\mu\overline{\psi}\gamma_\mu\gamma_5\psi\right] + 6b\Phi_\mu\Phi_\mu\}\; d^4x. \tag{4.43}$$

If we vary S with respect to Φ, we find

$$\begin{aligned} \Phi_\mu &= \frac{1}{24b}\overline{\psi}\gamma_\mu\gamma_5\psi \\ R &= -\frac{1}{4b}\left(\overline{\psi}\gamma_\mu\gamma_5\psi\right)\left(\overline{\psi}\gamma_\mu\gamma_5\psi\right). \end{aligned} \tag{4.44}$$

Now, the variation of S with respect to Ψ gives the following nonlinear equation, in which we recognize the Heisenberg equation (Borne, Lochak, & Stumpf, 2001), up to the coefficient

$$\gamma_\mu \partial_\mu \psi - \frac{1}{48b}\left(\overline{\psi}\gamma_\mu\gamma_5\psi\right)\gamma_\mu\gamma_5\psi = 0. \tag{4.45}$$

In so doing, *we come back once more to the monopole,* but now *in the nonlinear case* because Eq. (4.45) is a particular case of Eq. (4.2), by virtue of Eqs. (2.23), (2.24), and (4.44), which gives

$$R = \frac{1}{4b}\left(\omega_1^2 + \omega_1^2\right). \tag{4.46}$$

It means that the fundamental chiral invariant $\left(\omega_1^2 + \omega_1^2\right)$ that we defined, apart from a constant factor, is the curvature of the twisted space created by the self-action of the monopole, expressed in the equation by the identification of the torsion with the total curvature in Eq. (4.36). This confirms the link between our monopole and a torsion of the space.

CHAPTER 5

The Dirac Equation on the Light Cone. Majorana Electrons and Magnetic Monopoles

5.1 INTRODUCTION. HOW THE MAJORANA FIELD APPEARS IN THE THEORY OF A MAGNETIC MONOPOLE

In the first chapters of this book, we have developed the theory of a massless linear monopole, the quantized magnetic charge of which generalizes the Dirac formula. The neutrino appears as the fundamental zero state of the magnetic charge. The monopole is massless because the linear Dirac mass term would violate the *chiral gauge* invariance $\Psi \rightarrow \exp\left(i\frac{g}{\hbar c}\gamma_5\phi\right)\Psi$, which ensures the conservation of magnetism.

Nevertheless, in Chapter 4, we gave a generalization [Eq. (4.2)] of the linear equation [Eq. (2.16)], owing to the introduction of a *nonlinear mass term,* which is invariant with respect to the chiral gauge. There is an infinite family of such mass terms depending on an arbitrary function of a *chiral invariant* that is equal (up to a constant factor) to the space curvature.

Now we shall reexamine the problem of mass in another way. We shall consider *the Dirac equation on the relativistic light cone*, which gives a generalization of the *Majorana condition*. This result was achieved in Lochak (1987a,b, 1992, 2004). The main idea is that the *Majorana condition*, which reduces the Dirac equation to an abbreviated form, will be replaced by the condition that *the chiral invariant equals zero*, which is equivalent to writing the Dirac equation on the relativistic light cone if we define the light cone by the condition that the electric current (i.e., the velocity of the particle) is isotropic:

$$J_\mu J_\mu = 0, \tag{5.1}$$

However, by virtue of the algebraic relations [Eq. (2.24)]:

$$-J_\mu J_\mu = \Sigma_\mu \Sigma_\mu = \omega_1^2 + \omega_2^2 \ \left[= 4\left(\xi^+\eta\right)\left(\eta^+\xi\right) \text{ in the Weyl representation} \right].$$

Thus, Eq. (5.1) means that the chiral invariant equals zero on the light cone:

$$\rho = \sqrt{\omega_1^2 + \omega_2^2} = 0 \tag{5.2}$$

It must be noted that this definition is compatible with the conservation of electricity and magnetism because ρ is invariant under the ordinary gauge and the chiral gauge. Let us now consider the equations of the magnetic monopole (given in Chapter 4), with a nonlinear mass term.

So we have, in the Dirac representation:

$$\gamma_\mu \left(\partial_\mu - \frac{g}{\hbar c} \gamma_5 B_\mu \right) \Psi + \frac{1}{2} \frac{m(\rho)c}{\hbar} (\omega_1 - i\gamma_5 \omega_2) = 0. \tag{5.3}$$

And then, in the Weyl representation:

$$\left.\begin{aligned} &\frac{1}{c}\partial_t \xi - \mathbf{s}.\nabla\xi - i\frac{g}{\hbar c}(W + \mathbf{s}.\,\mathbf{B})\xi + i\frac{m(2|\xi^+\eta|)c}{\hbar}(\eta^+\xi)\ \ \eta = 0 \\ &\frac{1}{c}\partial_t \eta + \mathbf{s}.\nabla\eta + i\frac{g}{\hbar c}(W - \mathbf{s}.\,\mathbf{B})\eta + i\frac{m(2|\xi^+\eta|)c}{\hbar}(\xi^+\eta)\ \ \xi = 0 \end{aligned}\right. \quad \left\{B_\mu = (-i\mathbf{B}, W)\right\}. \tag{5.4}$$

These equations are invariant with respect to the chiral gauge transformation, and they represent a magnetic monopole. It was shown in Chapter 4 that the solutions of such equations as Eqs. (5.3) and (5.4) are divided into bradyon states (slower than light), tachyon states (faster than light), and luxon states (at the speed of light).

Just like the linear equations of the monopole, these nonlinear equations admit a nonlinear neutrino as a particular case for a zero charge $g = 0$, which means that such a nonlinear neutrino must have the same three states as the nonlinear monopole: bradyon, tachyon, and luxon. This hypothesis was previously formulated in another frame by Mignani and Recami (1975) and Recami and Mignani (1976).

Now the luxon state corresponds to the cancellation of the mass terms in Eqs. (5.3) and (5.4), which are thus reduced to the linear equations [Eqs. (2.16) and (3.4)]. But here, it does not mean a simple elimination of the mass term by the annihilation of a mass coefficient because m is not a simple coefficient, but a function. So that means a nonlinear condition on the wave functions:

$$\rho = 0 \;\Rightarrow\; \omega_1 = \omega_2 = 0 \;\Rightarrow\; \xi^+ \eta = 0. \tag{5.5}$$

The cancellation of the nonlinear term under the condition [Eq. (5.5)] does not imply the cancellation of the wave. The condition [Eq. (5.5)] is not exactly equivalent to the Majorana condition (Majorana, 1937; McLennan, 1957), which reads as $\psi = \psi_c$ $(\psi_c = \psi_{charge\ \ conjugated})$. Rather, it gives a slightly more general condition (Lochak, 1985):

$$\psi = e^{2i\frac{e}{\hbar c}\theta}\gamma_2\psi^* = e^{2i\frac{e}{\hbar c}\theta}\psi_c \;\Rightarrow\; \xi = e^{2i\frac{e}{\hbar c}\theta} is_2\eta^*, \quad \eta = -e^{2i\frac{e}{\hbar c}\theta} is_2\xi^*, \tag{5.6}$$

where $\theta(x, t)$ is an arbitrary phase (the coefficient $2e/\hbar c$ will be useful later).

In other words, the ψ state defined by Eq. (5.6) is its own charge-conjugate, but up to an arbitrary phase: this is almost the Majorana condition, which gives not exactly the Majorana-abbreviated equation. Later, we shall consider an equation that will not be abbreviated from the linear Dirac equation of the electron, but from the nonlinear equation of the monopole.

The fact that such a condition arises from the monopole theory leads us to explore it more precisely. Since the abbreviated Majorana equation was already suggested as a possible equation for the neutrino, we can ask: why would this not be the case for a magnetic monopole?

Nevertheless, for now we shall consider not the magnetic case, but the electric one. And we want to issue an initial warning: Do not be disappointed that we will be looking at the electric case for a longer time than the magnetic one. The reason for this is that, the magnetic case is far much complicated than the electric one, and that the last is interesting in itself. And it is not so elementary—and not only that, it is illuminating for our subject.

5.2 THE ELECTRIC CASE: LAGRANGIAN REPRESENTATION AND GAUGE INVARIANCE OF THE MAJORANA FIELD

Several authors (e.g., McLennan, 1957; Case, 1957; Berestetsky, Lifschitz, & Pitaevsky, 1972) have written about the problem of a Lagrangian representation of the Majorana field, and they concluded that such a representation is impossible. We shall see that that is wrong, but it is interesting to see where the difficulty is.

Using Eq. (2.11) and the Majorana condition, $\psi = \psi_c$, for an electrically charged particle in the presence of an electomagnetic field, the Majorana equation may be written as

$$\gamma_\mu \left(\partial_\mu - \frac{ie}{\hbar c} A_\mu \right) \psi + \frac{m_0 c}{\hbar} \psi_c = 0. \tag{5.7}$$

If we try to find a Lagrangian for such an equation directly, it must contain a term like the following:

$$\overline{\psi}\psi_c = \psi^+ \gamma_4 \gamma_2 \psi^*. \tag{5.8}$$

But we have, on the other hand:

$$\begin{aligned} \gamma_k &= i\alpha_4\alpha_k \ \ (k = 1, 2, 3), \ \ \gamma_4 = \alpha_4 \\ \alpha_k = \begin{bmatrix} 0 & s_k \\ s_k & 0 \end{bmatrix}, \ \alpha_4 &= \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}; \ \ (s_k = \text{Pauli matrices}). \end{aligned} \tag{5.9}$$

Introducing these expressions into Eq. (5.8), we have $\overline{\psi}\psi_c = 0$, and the corresponding term disappears from the Lagrangian, which is precisely the difficulty. But we shall proceed in another way: we consider the Majorana field as a constrained state of the Dirac field and express this constraint under the form of Eq. (5.5). Thus, we define the *Majorana Lagrangian* as a Dirac Lagrangian L_D, to which we add a constraint term with a Lagrange parameter λ:

$$L_M = L_D + \frac{\lambda}{2}\left(\omega_1^2 + \omega_2^2\right). \tag{5.10}$$

ω_1 and ω_2 are taken from Eq. (2.18), so that the variation of L_M, with respect to ψ, gives

$$\gamma_\mu \left(\partial_\mu - \frac{ie}{\hbar c} A_\mu \right) \psi + \frac{m_0 c}{\hbar} \psi + \lambda(\omega_1 - i\omega_2\gamma_5)\psi = 0. \tag{5.11}$$

This equation looks like our nonlinear equation [discussed in Chapter 4 and Lochak (1984, 1987a,b)], but here we have a mass term and an electric potential instead of the magnetic potential. In this form, the equation was found by Hermann Weyl (1950) and rediscovered later by other authors. The aim of Weyl (related to general relativity) was very different from ours.

Now, we vary the Lagrangian L_M [Eq. (5.10)] with respect to λ, which gives, using Eq. (5.6):

$$\gamma_\mu\left(\partial_\mu - \frac{ie}{\hbar c}A_\mu\right)\psi + \frac{m_0 c}{\hbar}e^{2i\frac{e}{\hbar c}\theta}\psi_c = 0 \tag{5.12}$$

It is the Majorana equation [Eq. (5.7)] with an *arbitrary* phase θ. We could write $\theta = 0$ in order to find Eq. (5.7), but that would be a bad idea because this phase is important: owing to this phase, Eq. (5.12) is *gauge-invariant,* while Eq. (5.7) is not. (By the way, nobody was worried about gauge-invariance)

In this case, the gauge invariance of Eq. (5.12) is a trivial consequence of the invariance of the Lagrangian [Eq. (5.10)]. But the invariance of Eq. (5.12) also can be directly demonstrated after the transformation:

$$\psi \to e^{i\frac{e}{\hbar c}\varphi}\psi, \quad A_\mu \to A_\mu - \partial_\mu\varphi, \quad \theta \to \theta + \varphi. \tag{5.13}$$

Now—and only now—the phase θ may be absorbed in the gauge and disappear. Therefore, we must first choose the gauge and only then cancel θ to find Eq. (5.7).

Therefore, the Majorana equation cannot be considered as independent: it is only the equation of a particular state (defined by a Lagrange multiplier) of the Dirac equation of the electron. And it is not gauge-invariant: only Eq. (5.11) is invariant. Nevertheless, we shall see that the Majorana equation may be considered itself, but this second interpretation is not equivalent to the preceding one.

5.3 TWO-COMPONENT ELECTRIC EQUATIONS. SYMMETRY AND CONSERVATION LAWS

Now, owing to Eq. (3.4), we find the Weyl representation of the class of the definite solutions of the Dirac equation:

$$(\pi_0 + \boldsymbol{\pi}.\mathbf{s})\xi - im_0 c e^{\frac{2ie}{\hbar c}\theta} s_2 \xi^* = 0, \tag{5.14a}$$

$$(\pi_0 - \boldsymbol{\pi}.\mathbf{s})\eta + im_0 c e^{2i\frac{e}{\hbar c}\theta} s_2 \eta^* = 0, \tag{5.14b}$$

$$\pi_0 = \frac{1}{c}\left(i\hbar\frac{\partial}{\partial t} + eV\right), \ \pi = \left(-i\hbar\nabla + \frac{e}{c}\mathbf{A}\right), \ \{A_\mu = (\mathbf{A}, iV)\}. \quad (5.15)$$

Eq. (5.14) is manifestly C-, P-, and T-invariant, but it is interesting to verify this property directly. Elementary calculations show, indeed, that the system [Eq. (5.14)] remains invariant by the following transformations using the Curie laws or those deduced from them (as covered in Chapter 4 and Poincaré, 1896):

$$\begin{aligned}
&(C): \ i \to -i, \quad e \to -e, \quad \xi \to e^{2i\frac{e}{\hbar c}\theta} is_2\eta^*, \quad \eta \to -e^{2i\frac{e}{\hbar c}\theta} is_2\xi^* \\
&(P): \ \mathbf{x} \to -\mathbf{x}, \quad \mathbf{A} \to -\mathbf{A}, \quad \xi \to i\eta \quad \eta \to -i\xi \\
&(T): \ e \to -e, t \to -t, V \to -V, \eta \to s_2\xi^*, \xi \to -s_2\eta^*.
\end{aligned} \quad (5.16)$$

The P transformation can be written in another way:

$$(P): \ \mathbf{x} \to -\mathbf{x}, \quad \mathbf{A} \to -\mathbf{A}, \quad \xi \leftarrow\rightarrow \eta, \quad \theta \to \theta + \frac{\pi}{2}\frac{\hbar c}{e}. \quad (5.17)$$

And the gauge transformation takes the following form:

$$\xi \to e^{i\frac{e}{\hbar c}\varphi}\xi, \ \ \eta \to e^{i\frac{e}{\hbar c}\varphi}\eta, \ \ \mathbf{A} \to \mathbf{A} - \nabla\varphi, \ \ V \to V + \frac{1}{c}\frac{\partial\varphi}{\partial t}, \ \ \theta \to \theta + \varphi. \quad (5.18)$$

It can be verified that the system [Eq. (5.14)] remains invariant under Eq. (5.18), which entails the conservation [Eq. (3.7)] of the chiral currents. It is important to note this conservation because it is true for a magnetic monopole (see Chapter 3), and here we see that it is also true for the solutions of the Dirac equation in the abbreviated case of an electron, restricted by the constraint $\rho = \sqrt{\omega_1^2 + \omega_2^2} = 0$ [Eq. (5.5)]. But this splitting into two equations is not applicable in the general case of the Dirac equation, which conserves only the electric current (the sum of the chiral currents), but not the magnetic current (see Chapter 3).

In the abbreviated electric case, the electric current is isotropic, the solutions of the Dirac equation are on the light cone, and the magnetic current disappears. Given that the expressions in Eq. (5.14) are split by the condition $\xi^+\eta = 0$, we can restrict ouselves, to only one of them—say the first one—and consider it in itself. This restricted equation is a chiral state of the electron. The second expression of Eq. (5.14) is the chiral conjugate of the first one, which means, owing to Eq. (5.16), that the image in a mirror is the time inverse of the first expression of Eq. (5.14).

5.4 THE CHIRAL STATE OF THE ELECTRON IN AN ELECTRIC COULOMB FIELD

Majorana considered that the equality $\psi = \psi_c$, introduced in the Dirac equation, gives something similar to a joint theory of the electron and the positron. But this is not the case because the preceeding equality is only a constraint imposed to the electron. Nevertheless, we have found a hybrid state: a kind of mixture of the electron and the positron. To show this, we shall solve the first expression of Eq. (5.14) in an electric coulomb field by introducing the following expressions:

$$eV = -\frac{e^2}{r}, \quad \mathbf{A} = 0, \quad \theta = \frac{\pi}{4}\frac{\hbar c}{e}. \tag{5.19}$$

These give the following equation:

$$\left[\frac{1}{c}\left(i\hbar\frac{\partial}{\partial t} - \frac{e^2}{r}\right) - i\hbar \mathbf{s}.\nabla\right]\xi + m_0 c s_2 \xi^* = 0. \tag{5.20}$$

The difficulty obviously lies in the complex conjugated ξ^*. So let us introduce the spherical functions with spin (Kramers, 1964; Bohm, 1960; Akhiezer & Berestetsky, 1965):

$$\Omega_\ell^m(+) = \begin{bmatrix} \left(\frac{\ell+m}{2\ell+1}\right)^{\frac{1}{2}} Y_\ell^{m-1} \\ \left(\frac{\ell-m+1}{2\ell+1}\right)^{\frac{1}{2}} Y_\ell^m \end{bmatrix}, \quad \Omega_\ell^m(-) = \begin{bmatrix} \left(\frac{\ell-m+1}{2\ell+1}\right)^{\frac{1}{2}} Y_\ell^{m-1} \\ -\left(\frac{\ell+m}{2\ell+1}\right)^{\frac{1}{2}} Y_\ell^m \end{bmatrix}, \tag{5.21}$$

in which Y_ℓ^m are the Laplace spherical functions ($\ell = 0, 1, 2, \ldots$; $\mathrm{m} = -\ell, -\ell+1, \ldots, \ell-1, \ell$):

$$Y_l^m(\theta,\varphi) = \frac{(-1)^m}{2^l l!}\left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}}\left(\frac{(l+m)!}{(l-m)!}\right)^{\frac{1}{2}}\frac{e^{im\varphi}}{\sin^l\theta}\frac{d^{l-m}}{d\theta^{l-m}}sin^{2l}\theta. \tag{5.22}$$

Now, we have the following equalities (for more information, see Appendix A of this chapter):

$$\begin{aligned}
&\mathbf{s.n}\ \Omega^m_{l-1}(+) = \Omega^m_l(-);\quad \mathbf{s.n}\ \Omega^m_l(-) = \Omega^m_{l-1}(+)\\
&\mathbf{s.n}\ s_2\ \Omega^{*m}_{l-1}(+) = i(-1)^{m+1}\Omega_l^{-m+1}(-)\\
&\mathbf{s.n}\ s_2\ \Omega^{*m}_l(-) = i(-1)^m\Omega_{l-1}^{-m+1}(+)
\end{aligned}\tag{5.23}$$

$$\begin{aligned}
&\mathbf{n} = \frac{\mathbf{r}}{r};\quad x = r\cos\varphi\sin\theta,\quad y = r\sin\varphi\sin\theta,\quad z = r\cos\theta\overrightarrow{n} = \frac{\mathbf{r}}{r};\\
&x = r\cos\varphi\sin\theta,\quad y = r\sin\varphi\sin\theta,\quad z = r\cos\theta.
\end{aligned}\tag{5.24}$$

We look for a solution of Eq. (5.20) of the following form:

$$\xi = \sum_m F^m_{\ell-1}(t,r)\,\Omega^m_{\ell-1}(+) + \sum_{m'} G^{m'}_\ell(t,r)\,\Omega^{m'}_\ell(-).\tag{5.25}$$

But it is impossible to separate the variables t and r immediately. It is only possible to separate the angular variables φ and θ. Following a classical procedure in the Dirac theory (Kramers, 1964; Akhiezer & Berestetsky, 1965), we introduce Eq. (5.25) into Eq. (5.20), multiplying the left side by $\mathbf{s.n}$. Owing to Eq. (5.23), we find

$$\begin{aligned}
&\frac{1}{c}\left(i\hbar\frac{\partial}{\partial t} - \frac{e^2}{r}\right)\left[F^m_{\ell-1}\ \Omega^m_\ell(-) + \sum_{m'} B^{m'}_\ell\ \Omega^{m'}_{\ell-1}(+)\right]\\
&= i\hbar\mathbf{s.n}\ \mathbf{s}.\nabla\left[\sum_m F^m_{\ell-1}\Omega^m_{\ell-1}(+) + \sum_{m'} B^{m'}_\ell\ \Omega^{m'}_\ell(-)\right]\\
&-im_0c\left[\sum_m (-1)^{m+1}F^{*m}_{\ell-1}\Omega_\ell^{-m+1}(-) + \sum_{m'}(-1)^{m'}B^{*m'}_\ell\Omega_\ell^{-m'+1}(+)\right].
\end{aligned}\tag{5.26}$$

The right-hand side is simplified owing to the classical relations:

$$\mathbf{s.n}\ \mathbf{s}.\nabla = \frac{\partial}{\partial r} - \frac{1}{r}\mathbf{s}.\Lambda,\tag{5.27}$$

where Λ is the orbital moment:

$$\Lambda = -i\mathbf{r}\times\nabla.\tag{5.28}$$

Now, we have other relations as follows (see Appendix B in this chapter):

$$\begin{aligned}
&\mathbf{s}.\Lambda\ \Omega^m_{\ell-1}(+) = (\ell-1)\ \Omega^m_{\ell-1}(+)\\
&\mathbf{s}.\Lambda\Omega^m_\ell(-) = -(\ell+1)\ \Omega^m_\ell(-),
\end{aligned}\tag{5.29}$$

so that, taking into account the fact that $\Omega_\ell^m(\pm)$ are orthonormal, we deduce from Eq. (5.26) the following system from which the angles are eliminated:

$$\begin{aligned}\left(\frac{1}{c}\frac{\partial}{\partial t}+i\frac{\alpha}{r}\right)F_{\ell-1}^m &= \left(\frac{\partial}{\partial r}+\frac{1+\ell}{r}\right)B_\ell^m+\chi(-1)^m F_{\ell-1}^{*-m+1}\\ \left(\frac{1}{c}\frac{\partial}{\partial t}+i\frac{\alpha}{r}\right)B_\ell^m &= \left(\frac{\partial}{\partial r}+\frac{1-l}{r}\right)F_{\ell-1}^m-\chi(-1)^m B_\ell^{*-m+1}\end{aligned} \tag{5.30}$$

$$m=-l,-l+1,\dots,l-1,l,\quad \alpha=\frac{e^2}{\hbar c},\quad \chi=\frac{m_0 c}{\hbar}. \tag{5.31}$$

In a subsequent step, we take the complex conjugate form of Eq. (5.30), changing $m \to -m+1$:

$$\begin{aligned}\left(\frac{1}{c}\frac{\partial}{\partial t}-i\frac{\alpha}{r}\right)F_{l-1}^{*-m+1} &= \left(\frac{\partial}{\partial r}+\frac{1+l}{r}\right)B_l^{*-m+1}-\chi(-1)^m F_{l-1}^m\\ \left(\frac{1}{c}\frac{\partial}{\partial t}-i\frac{\alpha}{r}\right)B_l^{*-m+1} &= \left(\frac{\partial}{\partial r}+\frac{1-l}{r}\right)F_{l-1}^{*-m+1}+\chi(-1)^m B_l^m.\end{aligned} \tag{5.32}$$

We combine Eqs. (5.30) and (5.32), introducing the new functions:

$$\begin{aligned}\frac{P_{\ell-1}^m(r)}{r}e^{(-1)^m i\omega t} &= F_{\ell-1}^m+(-1)^m F_{\ell-1}^{*-m+1};\\ \frac{Q_{\ell-1}^m(r)}{r}e^{(-1)^m i\omega t} &= F_{\ell-1}^m-(-1)^m F_{\ell-1}^{*-m+1};\\ \frac{R_\ell^m(r)}{r}e^{(-1)^m i\omega t} &= B_\ell^m+(-1)^m B_\ell^{*-m+1};\\ \frac{S_\ell^m(r)}{r}e^{(-1)^m i\omega t} &= B_\ell^m-(-1)^m B_\ell^{*-m+1};\end{aligned} \tag{5.33}$$

with

$$Q_{l-1}^m=(-1)^{m+1}P_{l-1}^{*-m+1};\quad S_l^m=(-1)^{m+1}R_l^{*-m+1}. \tag{5.34}$$

With the definition in Eq. (5.34), the notations [Eq. (5.33)] are invariant under complex conjugation and $m \to -m+1$. Summing and subtracting Eqs. (5.30) and (5.32), we find a first-order system in r (see Ince, 1956):

$$r\frac{dX}{dr}=(M+Nr)X; \tag{5.35}$$

$$X = \begin{bmatrix} P^m_{\ell-1}(r) \\ Q^m_{\ell-1}(r) \\ R^m_{\ell}(r) \\ S^m_{\ell}(r) \end{bmatrix}; \quad M = \begin{bmatrix} \ell & 0 & 0 & i\alpha \\ 0 & \ell & i\alpha & 0 \\ 0 & i\alpha & -\ell & 0 \\ i\alpha & 0 & 0 & -\ell \end{bmatrix};$$

$$N = \begin{bmatrix} 0 & 0 & i\dfrac{\omega'}{c} & -\chi \\ 0 & 0 & \chi & i\dfrac{\omega'}{c} \\ i\dfrac{\omega'}{c} & \chi & 0 & 0 \\ -\chi & i\dfrac{\omega'}{c} & 0 & 0 \end{bmatrix}; \quad \omega' = (-1)^m \omega. \tag{5.36}$$

The matrix N is diagonalized by

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & \dfrac{\omega'}{\mu c} & i\dfrac{\chi}{\mu} \\ 0 & 1 & -i\dfrac{\chi}{\mu} & \dfrac{\omega'}{\mu c} \\ 1 & 0 & -\dfrac{\omega'}{\mu c} & -i\dfrac{\chi}{\mu} \\ 0 & 1 & i\dfrac{\chi}{\mu} & -\dfrac{\omega'}{\mu c} \end{bmatrix}; \quad \mu = \sqrt{\frac{\omega^2}{c^2} - \chi^2}. \tag{5.37}$$

Introducing the new variable

$$Y = SX, \tag{5.38}$$

Eq. (5.35) takes the following form:

$$r\frac{dY}{dr} = \left\{ i\mu r \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + l \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} + \frac{\alpha}{\mu} \begin{bmatrix} \dfrac{\omega'}{c} s_1 + i\chi s_3 & 0 \\ 0 & -\dfrac{\omega'}{c} s_1 - i\chi s_3 \end{bmatrix} \right\} Y. \tag{5.39}$$

Here, μ is defined in Eq. (5.37), I is the unit matrix of the second order, and s_1, s_3 are Pauli matrices. We shall now diagonalize Eq. (5.39), changing the functions once more:

$$Z = \begin{bmatrix} V & 0 \\ 0 & s_1 V \end{bmatrix} Y; \quad V = \left[\frac{\omega'}{2\mu c}\right]^{\frac{1}{2}} \begin{bmatrix} \left[\dfrac{\omega'/c}{\mu - i\chi}\right]^{\frac{1}{2}} & \left[\dfrac{\mu - i\chi}{\omega'/c}\right]^{\frac{1}{2}} \\ \left[\dfrac{\omega'/c}{\mu + i\chi}\right]^{\frac{1}{2}} & -\left[\dfrac{\mu + i\chi}{\omega'/c}\right]^{\frac{1}{2}} \end{bmatrix}. \tag{5.40}$$

V is chosen such that

$$V\left(\frac{\omega'}{c} s_1 + i\chi s_3\right) V^{-1} = \mu s_3. \tag{5.41}$$

The equation takes a new form:

$$r\frac{dZ}{dr} = \left\{ i\mu r \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \ell \begin{bmatrix} 0 & s_1 \\ s_1 & 0 \end{bmatrix} + i\alpha \begin{bmatrix} s_3 & 0 \\ 0 & s_3 \end{bmatrix} \right\} Z, \tag{5.42}$$

and by iteration, we find

$$\left[r\frac{d}{dr}\right]^2 Z = \left\{ -\mu^2 r^2 + \mu r \left(i \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} - 2\alpha \begin{bmatrix} s_3 & 0 \\ 0 & s_3 \end{bmatrix} \right) + \ell^2 - \alpha^2 \right\} Z. \tag{5.43}$$

All the matrices are diagonalized, and we find four independent equations for the components of Z:

$$\left[r\frac{d}{dr}\right]^2 Z_n = \left[-\mu^2 r^2 + \mu r (i\varepsilon - 2\alpha\varepsilon') + \ell^2 - \alpha^2 \right] Z_n \quad (n = 1, 2, 3, 4) \tag{5.44}$$

$$\varepsilon = 1, 1, -1, -1; \quad \varepsilon' = 1, -1, -1, 1 \quad (\text{for} \quad n = 1, 2, 3, 4). \tag{5.45}$$

At this point, let us have the following:

$$r = \frac{i\rho}{2\mu}, \quad W_n = \rho^{\frac{1}{2}} Z_n. \tag{5.46}$$

Eq. (5.44) becomes, neglecting the suffix n,

$$\frac{d^2 W}{d\rho^2} + \left[-\frac{1}{4} + \frac{\frac{\varepsilon}{2} - i\alpha\varepsilon'}{\rho} + \frac{\frac{1}{4} + \alpha^2 - l^2}{\rho^2} \right] W = 0. \tag{5.47}$$

This is a Whittaker equation (Ince, 1956; Whittaker & Watson, 1958). The following coefficients are denoted here by k and m, keeping the classical notation for $W_{k,m}$. They are not to be confused with the previous other indices:

$$k = \frac{\varepsilon}{2} - i\alpha\varepsilon', \quad m = \sqrt{l^2 - \alpha^2}. \tag{5.48}$$

Thus, we can take the following Whittaker functions as radial functions, provided that they are square-integrable at the origin:

$$W_{k,m}(\rho) = W_{\frac{\varepsilon}{2} - i\alpha\varepsilon', \sqrt{l^2-\alpha^2}}\left(-2i\mu r\right). \tag{5.49}$$

However, in the vicinity of the origin, a regular solution of Eq. (5.47) may be written in the following form (Ince, 1956; Whittaker & Watson, 1958), taking into account Eqs. (5.46) and (5.48):

$$\left|W_{k,m}\right| = 2\mu r^{\frac{1}{2}+m}\Big(1 + O(r)\Big) \tag{5.50}$$

It must be noted that the same coefficient m appears in all the composants W_n, and thus in Z_n in Eq. (5.44); therefore, with the changes in Eqs. (5.46), (5.40), (5.38), (5.34), and (5.25), we can assert that

$$\xi^{+}\xi \approx r^2(m-1) \quad (\text{in the vicinity of } r = 0). \tag{5.51}$$

So the value of m [Eq. (5.48)] shows that $\xi^{+}\xi$ is always integrable at the origin because $l = 0, 1, 2\ldots$ But even more interesting is the behavior at the infinity. From standard formulas, we have (Whittaker & Watson, 1958)

$$W_{k,m}(\rho) = e^{-\frac{1}{2}\rho}\rho^{k}\Big(1 + O\big(\rho^{-1}\big)\Big) \quad \text{if} \quad \Big|\mathrm{Arg}(-\rho)\Big| < \pi. \tag{5.52}$$

The condition of validity is satisfied because $\rho = -2i\mu r$, by virtue of Eq. (5.46), so that, owing to Eq. (5.48):

$$W_{k,m}(\rho) = 2\mu r^{\frac{\varepsilon}{2}}\big(1 + O(r^{-1}\big) \quad [\varepsilon = \pm 1, \text{ as in Eq. (5.44)}]. \tag{5.53}$$

If we now consider the change of functions $\xi^{+}\xi$, we encounter some difficulty. In $r \approx 0$, we had the same exponents in Eq. (5.50) for all the components W and Z, but now the situation is different with the exponent $\frac{\varepsilon}{2}$ in Eq. (4.35). Using Eqs. (5.46), (5.40), (5.38), (5.34), and (5.25) once more, we find for $\xi^{+}\xi$ the following asymptotic form:

$$\xi^{+}\xi = \sum a_{nn'} r^{\varepsilon_n + \varepsilon_{n'} - 3} \quad (\text{for } r \to \infty), \tag{5.54}$$

where, according to Eq. (5.46), ε_n take the values $\varepsilon = \pm 1$ for the different components of Z, which leads to several conclusions, discussed next.

5.5 CONCLUSIONS FROM THE PHYSICAL BEHAVIOR OF A CHIRAL STATE OF A DIRAC ELECTRON (A MAJORANA ELECTRON), IN AN ELECTRIC COULOMBIAN FIELD

The asymptotic form [Eq. (5.54)] shows that $\xi^{+}\xi$ *would be integrable in the whole space only if*, in the sum of the second member of Eq. (5.54), ε_n is never equal to 1. The different values of ε_n and $\varepsilon_{n'}$ give terms with r^{-5} (for $\varepsilon_n + \varepsilon_{n'} = -2$); r^{-3} (for $\varepsilon_n + \varepsilon_{n'} = 0$); r^{-1} (for $\varepsilon_n + \varepsilon_{n'} = 2$).

Now, only the first type of term gives a convergent integral as $r \to \infty$. In order for the integral of $\xi^{+}\xi$ to converge, we must exclude the terms with $\varepsilon_n = 1$, which implies the annihilation of the components Z_1 and Z_2 in Eq. (5.42). But if we do this, we get $Z \equiv 0$ and the wave function disappears.

$\xi^{+}\xi$ is, thus, *never integrable* on the whole space. Therefore, the Majorana electron has no bound states in a Coulomb field: the spectrum is continuous and there are only ionized states. It must be noticed that the sign of α in Eq. (5.42) does not play just any role: the Majorana electron—more precisely, the Majorana state of the Dirac electron—has a diffusive behavior of the same type, independent of a positive or negative charge of the coulomb field.

It is easy to understand why this is so. In the state of ξ [Eq. (5.25)], which is associated with a value $\frac{l-1}{2}$ of the total kinetic momentum, the terms corresponding to the different values m have, according to Eq. (5.33), exponential factors $e^{(-1)^{m}\omega t}$, where ω is the energy such that ξ *is a superposition of states with positive and negative energies,* corresponding to the *electron or positron states*.

Thus, the Majorana theory is not a "simultaneous theory of the electron and of the positron". It is only a hybrid state of the Dirac electron, "which does not know" the sign of its electric charge. We understand why it cannot be in a bound state. But its diffusing states will be very different from the state of a fast, "normal" electron state because the wave functions are different from the wave functions of a Keplerian system in a ionized state.

To make this fact more understandable, we shall carry out the preceding calculation in the classical limit, and we shall see that all the trajectories are hyperbolic, as it might be guessed, but the hyperbolas are *not Keplerian*. And given that the classical limit does not know the quantum superposition, there are two kinds of hyperbolas corresponding respectively to the diffusion, in an attractive or a repulsive field.

5.6 THE GEOMETRICAL OPTICS APPROXIMATION OF THE STATES OF THE MAJORANA ELECTRON

Consider the general equation [the first expression of Eq. (5.14)] for ξ, with the definitions [Eq. (5.15)] and the electromagnetic gauge [Eq. (5.19)]. Now, in the first expression of Eq. (5.14), we introduce the following expression [$a(t,\mathbf{r})$ and $b(t,\mathbf{r})$ are new *spinors*]:

$$\xi = a(t,\mathbf{r})e^{-\frac{i}{\hbar}S(t,\mathbf{r})} + b(t,\mathbf{r})e^{+\frac{i}{\hbar}S(t,\mathbf{r})}. \tag{5.55}$$

Neglecting the $\hbar$ terms, we have the following equation:

$$\begin{aligned}&\left\{\left[\frac{1}{c}\left(\frac{\partial S}{\partial t}+eV\right)-\left(\nabla S-\frac{e}{c}\mathbf{A}\right).\mathbf{s}\right]a+m_0cs_2b^*\right\}e^{-\frac{i}{\hbar}S}\\&-\left\{\left[\frac{1}{c}\left(\frac{\partial S}{\partial t}-eV\right)-\left(\nabla S+\frac{e}{c}\mathbf{A}\right).\mathbf{s}\right]b-m_0cs_2a^*\right\}e^{+\frac{i}{\hbar}S}=0.\end{aligned} \tag{5.56}$$

For $\hbar\to 0$, the phases $\pm\frac{S}{\hbar}$ become infinitely fast, and, multiplying Eq. (5.56) by $e^{\frac{iS}{\hbar}}$ and $e^{\frac{-iS}{\hbar}}$, alternately, we find the geometrical optics approximation:

$$\begin{aligned}&\left[\frac{1}{c}\left(\frac{\partial S}{\partial t}+eV\right)-\left(\nabla S-\frac{e}{c}\mathbf{A}\right).\mathbf{s}\right]a+m_0cs_2b^*=0\\&\left[\frac{1}{c}\left(\frac{\partial S}{\partial t}-eV\right)+\left(\nabla S+\frac{e}{c}\mathbf{A}\right).\mathbf{s}\right]b-m_0cs_2a^*=0.\end{aligned} \tag{5.57}$$

Now we introduce a new spinor $\widehat{b}$:

$$\widehat{b} = s_2b^*. \tag{5.58}$$

Taking the complex conjugate of the second equation [Eq. (5.57)] multiplied on the left by s_2 (taking into account that s_2 is imaginary, which gives the plus sign in the second equation), one obtains for Eq. (5.57):

$$\begin{aligned}&\left[\frac{1}{c}\left(\frac{\partial S}{\partial t}+eV\right)-\left(\nabla S-\frac{e}{c}\mathbf{A}\right).\mathbf{s}\right]a+m_0c\widehat{b}=0\\&\left[\frac{1}{c}\left(\frac{\partial S}{\partial t}-eV\right)+\left(\nabla S+\frac{e}{c}\mathbf{A}\right).\mathbf{s}\right]\widehat{b}+m_0ca=0.\end{aligned} \tag{5.59}$$

Multiplying the first equation by the matrix before $\widehat{b}$ in the second equation, we get

$$\begin{gathered}\left\{\left[\frac{1}{c}\left(\frac{\partial S}{\partial t}-eV\right)+\left(\nabla S+\frac{e}{c}\mathbf{A}\right).\mathbf{s}\right]\left[\frac{1}{c}\left(\frac{\partial S}{\partial t}+eV\right)\right.\right.\\ \left.\left.-\left(\nabla S-\frac{e}{c}\mathbf{A}\right).\mathbf{s}\right]-m_0^2c^2\right\}a=0\end{gathered} \tag{5.60}$$

or

$$\left\{\begin{aligned}&\frac{1}{c}\left(\frac{\partial S}{\partial t}+eV\right)\left(\frac{\partial S}{\partial t}-eV\right)-\left(\nabla S+\frac{e}{c}\mathbf{A}\right)\left(\nabla S-\frac{e}{c}\mathbf{A}\right)-m_0^2c^2\\ &+\ 2\frac{e}{c}\left[V\nabla S+\frac{1}{c}\frac{\partial S}{\partial t}\mathbf{A}+i\nabla S\times\mathbf{A}\right].\mathbf{s}\end{aligned}\right\}a=0. \tag{5.61}$$

For $a\neq 0$, we must set equal to zero the determinant of the matrix, which gives a Hamilton-Jacobi equation that reads, for $\mathbf{A}=0$:

$$\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-(\nabla S)^2-\frac{e^2}{c^2}V^2-m_0^2c^2\right]^2-\frac{4e^2}{c^2}V^2(\nabla S)^2=0. \tag{5.62}$$

The factorization of the difference of two squares gives two equations that take the following form in the coulomb case:

$$\frac{1}{c}\left(\frac{\partial S}{\partial t}\right)^2-\left(\left|\nabla S\right|-\frac{\varepsilon e^2}{c}\frac{1}{r}\right)^2-m_0^2c^2=0\quad(\varepsilon=\pm 1). \tag{5.63}$$

We can see that the sign of the charge does not play any role because $\varepsilon=\pm 1$ not due to the charge, but to the factorization. And, still more important, these Hamilton-Jacobi equations are different from those that are found in the well-known problem of an electron in a coulomb field. In the latter case, we have the following equations with two signs $\varepsilon=\pm 1$ as well, but they are now due to the sign of the charge, and they correspond to two kinds of trajectories, ellipses, or hyperbolas:

$$\frac{1^2}{c}\left(\frac{\partial S}{\partial t}-\frac{\varepsilon e^2}{r}\right)^2-(\nabla S)^2-m_0^2c^2=0\quad(\varepsilon=\pm 1). \tag{5.64}$$

Now, if we introduce in Eq. (5.63) the decomposition

$$S=-Et+W, \tag{5.65}$$

we find

$$\frac{E^2}{c^2} - m_0^2 c^2 = \left[\left| \nabla W \right| - \frac{\varepsilon e^2}{c} \frac{1}{r} \right]^2, \tag{5.66}$$

from which it follows immediately that

$$E \;\geq\; m_0 c^2. \tag{5.67}$$

This means that there is not any bound state, and thus no closed trajectories. Actually, we have in Eq. (5.66) two equations:

$$(\nabla W)^2 = \frac{1}{c^2}\left(\sqrt{E^2 - m_0^2 c^4} + \frac{\varepsilon e^2}{r} \right)^2 \quad (\varepsilon = \pm 1), \tag{5.68}$$

and thus, in polar coordinates:

$$(\nabla W)^2 = \left(\frac{\partial W}{\partial r} \right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \varphi} \right)^2. \tag{5.69}$$

Now, if we write

$$W = J\varphi + f(r) \quad (J = \text{Const.}), \tag{5.70}$$

Eq. (5.68) is transformed into

$$f(r) = \int \left(A^2 + \frac{2B}{r} + \frac{C}{r^2} \right)^{\frac{1}{2}} dr, \tag{5.71}$$

with

$$A = \frac{1}{c}\sqrt{E^2 - m_0^2 c^4}, \quad B = \frac{A\varepsilon e^2}{c}, \quad C = \frac{e^4}{c^2} - J^2. \tag{5.72}$$

The discriminant in Eq. (5.71) is positive:

$$\Delta' = B^2 - A^2 C = A^2 J^2 \geq 0 \tag{5.73}$$

As a result, the roots are real:

$$\frac{1}{r} = \frac{A\left(\frac{\varepsilon e^2}{c} \pm J \right)}{J^2 - \frac{e^4}{c^2}} \quad (\varepsilon = \pm 1). \tag{5.74}$$

Suppose now that $J \neq 0$. Given that we are in the limit of a quantum problem, we can write

$$J \approx \hbar = \frac{e^2}{\alpha c} = 137\frac{e^2}{c} \gg \frac{e^2}{c} \tag{5.75}$$

This approximation is not essential, but it is convenient for what follows because we can write the positive root [Eq. (5.74)] in a simplified form:

$$\frac{1}{r^*} = \frac{A}{J} \quad \text{with} \ : \quad r^* = \frac{Jc}{\sqrt{E^2 - m_0^2 c^4}}. \tag{5.76}$$

So the trajectory defined by Eqs. (5.70) and (5.71) is now

$$\frac{\partial W}{\partial J} = \varphi_0 \to \varphi - \varphi_0 = J\int_{r^*}^{r} - \frac{1}{\left(A^2 + \frac{2B}{r} + \frac{C}{r^2}\right)^{\frac{1}{2}}} \frac{dr}{r^2}. \tag{5.77}$$

Taking $\varphi_0 = 0$, with the approximation [Eq. (5.76)], the equation of the trajectory becomes

$$\frac{1}{r} = \frac{e^2\sqrt{E^2 - m_0^2 c^4}}{J^2 c^2}\left(\varepsilon + \frac{Jc}{e^2}\cos\varphi\right) \quad (\varepsilon = \pm 1). \tag{5.78}$$

It is a *hyperbola* because by virtue of Eq. (5.75), its eccentricity is greater than 1:

$$\frac{Jc}{e^2} > 1. \tag{5.79}$$

It must be underscored that the hyperbolic character of the trajectory already had been determined by Eq. (5.67) and not only by the simplified form [i.e., Eq. (5.76)]. In conclusion, there are not any bound state as it had previously been noted, but do not forget that there are two possible types of trajectories because $\varepsilon = \pm 1$, the two signs corresponding to the two equations [Eq. (5.63)]. To wit:

- If $\varepsilon = +1$, the *concavity* of the trajectory is oriented to the central field and the motion is *attractive*.
- If $\varepsilon = -1$, the *convexity* of the trajectory is oriented to the central field and the motion is *repulsive*.

Therefore, in accordance with the quantum treatement, both cases are possible, whatever the charges and the central field might be.

It is interesting to compare these results with the classical case of a relativistic electron in a coulombian potential: we consider the classical equation

[Eq. (5.64)] again, introducing Eqs. (5.69) and (5.70), which gives an integral of the same form as Eq. (5.71):

$$f(r) = \int \left(A^2 + \frac{2B'}{r} + \frac{C}{r^2}\right)^{\frac{1}{2}} dr \tag{5.80}$$

$$A = \frac{1}{c}\sqrt{E^2 - m_0^2c^4}, \quad B' = \frac{E\varepsilon e^2}{c^2}, \quad C = \frac{e^4}{c^2} - J^2. \tag{5.81}$$

In the case of $E \geq m_0c^2$, in comparison with Eq. (5.72), one can see that the only coefficient B remains, while the factor A is substituted by E/c, which means the coincidence of these two cases for the limit $E \to m_0c^2$. But it must be noted that, in the preceding case, the condition $E \geq m_0c^2$ [Eq. (5.67)] was necessary, while here, in the classical case, it is only one of two possibilities because we could have $E < m_0c^2$, which would correspond to elliptic trajectories (i.e., bound states).

Taking the preceding calculation again with the constants [Eq. (5.81)], we find the trajectories as follows:

$$\frac{1}{r} = \frac{e^2E}{J^2c^2}\left(\varepsilon + c\frac{\sqrt{(E^2 - m_0^2c^4)J^2 + m_0^2c^2e^4}}{Ee^2}\cos\varphi\right). \tag{5.82}$$

This formula, which is good only for $E > m_0c^2$, differs from the classical formula only by the absence of the precession factor in the argument of the cosine, which we have neglected by virtue of Eq. (5.75), and the preceding approximation, which actually results in the replacement of C by $-J^2$. On the contrary, the approximation would not be valuable under the root sign in the expression of the eccentricity except if $E \gg m_0c^2$, which is the limit to which Eqs. (5.78) and (5.82) tend.

But the interesting case arises when $E - m_0c^2$ is small, because the eccentricity of the classical hyperbola depends on E and

$$c\frac{\sqrt{(E^2 - m_0^2c^4)J^2 + m_0^2c^4}}{Ee^2} \to 1 \quad \text{if} \quad E \to m_0c^2. \tag{5.83}$$

Thus, the classical *parabolic* trajectory results when $E \to m_0c^2$.

On the contrary, the eccentricity of the hyperbola [Eq. (7.78)] is independent of energy and, consequently, from the angle between the asymptotes, while

$$\frac{1}{p} = \frac{e^2\sqrt{E^2 - m_0^2c^4}}{J^2c^2} \to 0 \quad \text{if} \quad E \to m_0c^2. \tag{5.83'}$$

Therefore, the parameter approches infinity, while Eq. (5.82) shows that in the classical case, when $E \to m_0c^2$, the parameter tends toward a finite value. Consequently, for low energies, we find two different ways of behavior that could be experimentally distinguished, provided that one could create this strange, constrained state of the electron described by the Majorana field.

5.7 HOW COULD ONE OBSERVE A MAJORANA ELECTRON?

We have seen that in a coulomb field, at the geometrical optic approximation, the Majorana electron behaves either like a particle with a negative charge or like a particle with a positive charge, but it remains different from an electron or a positron because its motion is not Keplerian.

Nevertheless, this is only a problem of trajectories; that is, a problem of the rays of the wave given by the Jacobi equation. If we introduce the corresponding approximate expression of the action S in the expression of the wave function [Eq. (5.55)], we find an approximate solution of the equations [Eq. (5.57)].

Therefore, we shall find that, despite that trajectories seem to "choose" their charge (+ or −), the wave function evidently remains a superposition state of two waves with *opposite phases;* that is, waves with conjugated charges. Let us apply that concept to plane waves.

We write Eq. (5.55) with constant spinors a and b:

$$\xi = a \; e^{i(\omega t - \mathbf{k}.\mathbf{r})} + b \; e^{-i(\omega t - \mathbf{k}.\mathbf{r})}, \tag{5.84}$$

and we introduce Eq. (5.84) into the first expression in Eq. (5.14) with $V = A = 0$, and an angle θ, which is defined in Eq. (5.19). Analogous to the one of the § 5.6, a simple computation gives

$$\frac{\omega^2}{c^2} = k^2 + \frac{m_0^2c^2}{\hbar^2} \tag{5.85}$$

$$\xi = a \; e^{i(\omega t - \mathbf{k}.\mathbf{r})} - \frac{\hbar}{m_0 c}\left(\frac{\omega}{c} - \mathbf{s}.\mathbf{k}\right)s_2 a^* \; e^{-i(\omega t - \mathbf{k}.\mathbf{r})}. \tag{5.86}$$

This is a superposition of two waves with energies of opposite signs. But let us return to the Dirac equation; that is, the two expressions in Eq. (5.14) linked by Eq. (5.6), with the condition [Eq. (5.19)]. Therefore, it is not exactly the Majorana field but the Dirac field that is constrained by Eq. (5.5). In other words, it is the equation [Eq. (5.12)] with the value of Eq. (5.19) for the angle θ, and $A_\mu = 0$.

Now we must find the wave ψ, owing to Eq. (5.86) and

$$\eta = s_2 \xi^*, \quad \psi = \frac{1}{\sqrt{2}} \begin{bmatrix} \xi + \eta \\ \xi - \eta \end{bmatrix}. \tag{5.87}$$

We shall take $0z$ for the direction of propagation of the wave and

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \mathbf{k} = \{0, 0, k\}, \tag{5.88}$$

with a_1 and a_2 = components of a, in Eq. (5.86). So we find

$$\psi = \frac{1}{\sqrt{2}} (\psi_1 + \psi_2) \tag{5.89}$$

$$\psi_1 = a_1 \begin{bmatrix} 1 + \dfrac{\hbar}{m_0 c}\left(\dfrac{\omega}{c} + k\right) \\ 0 \\ 1 - \dfrac{\hbar}{m_0 c}\left(\dfrac{\omega}{c} + k\right) \\ 0 \end{bmatrix} e^{i(\omega t - kz)} - i a_1^* \begin{bmatrix} 0 \\ 1 + \dfrac{\hbar}{m_0 c}\left(\dfrac{\omega}{c} + k\right) \\ 0 \\ -\left[1 - \dfrac{\hbar}{m_0 c}\left(\dfrac{\omega}{c} + k\right)\right] \end{bmatrix} e^{-i(\omega t - kz)} \tag{5.90}$$

$$\psi_2 = a_2 \begin{bmatrix} 0 \\ 1 + \dfrac{\hbar}{m_0 c}\left(\dfrac{\omega}{c} - k\right) \\ 0 \\ 1 - \dfrac{\hbar}{m_0 c}\left(\dfrac{\omega}{c} - k\right) \end{bmatrix} e^{i(\omega t - kz)} + i a_2^* \begin{bmatrix} 1 + \dfrac{\hbar}{m_0 c}\left(\dfrac{\omega}{c} - k\right) \\ 0 \\ -\left[1 - \dfrac{\hbar}{m_0 c}\left(\dfrac{\omega}{c} - k\right)\right] \\ 0 \end{bmatrix} e^{-i(\omega t - kz)}. \tag{5.91}$$

Here, ψ is the superposition of two waves ψ_1 and ψ_2 with the constants a_1 and a_2. Each wave ψ_1 and ψ_2 depends on energy and helicity, which is easy to define because if Oz is the direction of propagation so that the spin is projected in the same direction, and

$$\sigma_3 = \begin{bmatrix} s_3 & 0 \\ 0 & s_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{5.92}$$

then we see the following:

1. $\psi_1$is a superposition of two waves with the same sign of helicity and charge (+ and −, respectively, for each wave).
2. ψ_2 is a superposition of two waves with opposite helicities and charges.

The relative phase of the components of ψ_1 or ψ_2 (i.e., $a_{1,2}$ and $a^*_{1,2}$) has no physical meaning because the constant θ in Eq. (5.12) or (5.14) is arbitrary. Now, for low energies,

$$|k| \ll \frac{\omega}{c}, \quad \frac{\omega}{c} = \frac{m_0 c}{\hbar}, \tag{5.93}$$

we have, in a first approximation:

$$\psi_1 = \begin{bmatrix} a_1 e^{i(\omega t - kz)} \\ -i a_1^* e^{-i(\omega t - kz)} \\ 0 \\ 0 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} i \ a_2 e^{-i(\omega t - kz)} \\ a_2^* e^{i(\omega t - kz)} \\ 0 \\ 0 \end{bmatrix}. \tag{5.94}$$

In conclusion, if we could "keep alive" (i.e., keep from destroying) two parallel-beam of electrons and positrons with the same energy and this polarization for a sufficiently long time, the definite couples would have the behavior of a Majorana electron. In particular, in a coulomb field, an electron in such a state would exhibit the strange behavior just described instead of following the classical Kepler laws.

5.8 THE EQUATION IN THE MAGNETIC CASE

We have recalled in Eq. (5.3) the general nonlinear equation of a magnetic monopole, and we know that the chiral gauge invariance is broken and the magnetic charge is no more conserved if we add a linear mass term (it is the reason for which the Dirac equation does not conserve the magnetic charge). As we know (as discussed in Chapter 3), the Majorana condition ensures the conservation of chiral currents, and thus of magnetism.

Such an equation is not really chiral gauge-invariant, but in this case, it admits a *subset of gauge-invariant solutions*. Now, remember that the chiral invariance is an invariance with respect to the rotations in the chiral plane $\{\omega_1, \omega_2\}$ (i.e., with respect to the rotations of an angle A), which can be obtained in two ways:

1. The first way is to introduce in the Lagrangian a mass term that depends only on the norm of the vector $\{\omega_1, \omega_2\}$; this was done until now, and it results in Eq. (5.3).
2. The second way is to add to the Lagrangian of the linear monopole an arbitrary mass term that is not necessarily chiral-invariant (as was the norm of $\{\omega_1, \omega_2\}$), but which is such that the obtained equation has a subset of solutions that annihilates the chiral invariant:

$$\rho = \left(\omega_1^2 + \omega_2^2\right)^{1/2} = 0. \tag{5.95}$$

Such solutions thus obey the generalized Majorana condition [Eq. (5.6)], which we write here in a simpler form:

$$\Psi = e^{i\theta}\gamma_2\psi^* = e^{i\theta}\psi_c. \tag{5.96}$$

Actually, we can put $\theta = 0$, as discussed later in this chapter. A priori, we could start from an arbitrary term of mass, but for simplicity, we shall choose the linear mass term of the Dirac equation. So now we can introduce, in the equation of the massless monopole, the mass term of Eq. (2.1) under the condition Eq. (5.95) or (5.96), which will be expressed by means of a Lagrange multiplier. Thus, we have the Lagrangian:

$$L = \overline{\Psi}\gamma_\mu\left[\partial_\mu\right]\Psi - g\hbar c\overline{\Psi}\gamma_\mu\gamma_5 B_\mu\Psi - m_0 c\hbar\overline{\psi}\psi + \lambda\left(\omega_1^2 + \omega_2^2\right), \tag{5.97}$$

from which, varying $\overline{\psi}$, we deduce the following equation, which looks like our nonlinear equation from Chapter 4, but with a linear term:

$$\gamma_\mu\left(\partial_\mu - g\hbar c\gamma_5 B_\mu\right)\Psi - m_0 c\hbar\psi + \lambda(\omega_1 - i\omega_2\gamma_5)\Psi = 0. \tag{5.98}$$

The difference between this equation and the equation of our nonlinear monopole is the presence of a linear mass term and of the constant λ instead of $m(\rho^2)$. But the linear term will be transformed, and the nonlinear term itself will disappear because we must vary L with respect to the Lagrange multiplier λ, in order to find Eq. (5.95). Thus, we have

$$\omega_1 = \omega_2 = 0, \tag{5.99}$$

which gives Eq. (5.97) and annihilates the λ term in Eq. (5.98). The Lagrange multiplier thus remains undetermined, since it does not appear in the equation. If we introduce Eq. (5.96), **we find the Majorana equation up to a phase factor** $e^{i\theta}$, with a magnetic interaction instead of an electric one:

$$\gamma_\mu\left(\partial_\mu - g\hbar c\gamma_5 B_\mu\right)\Psi - m_0 c\hbar e^{i\theta}\gamma_2\psi^* = 0. \tag{5.100}$$

It is a new, nonlinear equation of a magnetic monopole, different from the one found earlier. In the Weyl representation (discussed in Chapter 3), Eq. (5.100) splits into two equations that are formally separated, but are actually linked to each other:

$$\begin{aligned}\left(\pi_0^+ + \pi^+\cdot s\right)\xi - im_0 ce^{i\theta}s_2\xi^* &= 0\\ \left(\pi_0^- - \pi^-\cdot s\right)\eta + im_0 ce^{i\theta}s_2\eta^* &= 0,\end{aligned} \tag{5.101}$$

with the following definitions:

$$\begin{aligned}\pi_0^+ &= \frac{1}{c}\left(i\hbar\frac{\partial}{\partial t} + gW\right), \quad \pi^+ = -i\hbar\frac{\partial}{\partial t} + \frac{g}{c}\ \mathbf{B}\\ \pi_0^- &= \frac{1}{c}\left(i\hbar\frac{\partial}{\partial t} - gW\right), \quad \pi^- = -i\hbar\frac{\partial}{\partial t} - \frac{g}{c}\ \mathbf{B}.\end{aligned} \tag{5.102}$$

We can remark that, *in the electric case, we had only one operator* $\{\pi_0, \pi\}$, while *in the magnetic case, we have two operators: right and left*. Before examining Eq. (5.101), we must take a moment to specify some points concerning Eq. (5.98).

First, this equation was found a long time ago by Weyl (1950), only *for a free wave* (*i.e., without interaction),* and with another aim. For Weyl, the nonlinear term was not a Lagrange condition. Rather, it was a change of the Dirac equation, owing to which the nonlinear Weyl equation (contrary to the Dirac linear equation) has the property of keeping the same form, in general relativity, if it were expressed in metric form with an affine connection $\Gamma_{\mu\lambda\nu}$, depending on $g_{\mu\nu}$; or with coefficients $\Gamma_{\mu\lambda\nu}$, independent of $g_{\mu\nu}$.

In various forms, the same equation without interactions was later found again by several authors and reexamined from different points of view. Two papers are particularly interesting with respect to this problem:

1. The first (Rodichev, 1961) already has been discussed in Chapter 4. Just recall that it is based on a particular case of Eq. (5.98), where λ is an ordinary constant:

$$\gamma_\mu\partial_\mu\Psi + \lambda(\Omega_1 - i\Omega_2\gamma_5)\Psi = 0. \tag{5.103}$$

It was shown (both in Chapter 4 and Lochak, 1985e) that the chiral invariant is equal, up to a constant factor, to the total curvature. But in this case, the space is flat and the curvature is reduced to the torsion, so that when we show that the Majorana condition [Eq. (5.96)] is equivalent to the condition [Eq. (5.95)] it actually signifies that the Majorana condition annihilates the torsion of the space.

2. Now we give results due to A. Bachelot (1988a,b), who solved the global Cauchy problem for Eq. (5.103) without electromagnetic interaction, but with initial conditions that are not supposed to be small: they are only so to the extent that the chiral invariant $\rho = (\omega_1^2 + \omega_2^2)^{1/2}$ is small. In other words, it remains in the vicinity of the condition [Eq. (5.95)], which, as already established, is close to the generalized Majorana condition.

To prove his theorem, Bachelot first proved the following lemma, which is of great interest in itself:

- Consider the Dirac equation without interaction, but with a mass term M, possibly depending on space and time:

$$\gamma_\mu \partial_\mu \Psi + M\psi = 0 \tag{5.104}$$

Bachelot proved that if the chiral invariant $\rho = (\omega_1^2 + \omega_2^2)^{1/2}$ vanishes at a given instant in the whole space, it remains equal to zero later. It is easy to generalize the lemma of Bachelot in the presence of a *magnetic interaction,* and we shall directly formulate and prove it in this more general case.

- Given the equation

$$\gamma_\mu \left(\partial_\mu \Psi - g\hbar c \gamma_5 B_\mu\right)\Psi - m_0 c\hbar\psi = 0, \tag{5.105}$$

if at a given instant, the chiral invariant $\rho = (\Omega_1^2 + \Omega_2^2)^{1/2}$ (and so, the torsion of the space) vanishes in the whole space, il remains equal to zero. Bachelot starts from two conservation laws:

$$\partial_\mu \overline{\psi}\gamma_\mu\psi = 0, \ \ \partial_\mu \overline{\tilde{\psi}}\gamma_2\gamma_4\gamma_\mu\psi = 0 \ \ \left(\tilde{\psi} = \text{transposed } \psi\right). \tag{5.106}$$

The first law is the conservation of the Dirac current (i.e., of electricity). It must be noted that the chiral currents are not separately conserved, contrary to Eq. (3.7), because of the presence of a linear mass term in Eq. (5.105); but their sum is conserved as in the Dirac equation, and this sum is precisely the Dirac electric current that appers in Eq. (5.106).

The second law is the conservation of the crossed current between charge-conjugated states. Bachelot deduced it from Eq. (5.104), but it is also true for Eq. (5.105), with a magnetic interaction. On the contrary, the second conservative law would be wrong in the case of an ordinary Dirac equation with an electric interaction. Indeed, we get in this case:

$$\partial_\mu \tilde{\psi} \gamma_2 \gamma_4 \gamma_\mu \psi + iA_\mu \tilde{\psi} \gamma_2 \gamma_4 \gamma_\mu \psi = 0. \tag{5.107}$$

Now, if these two laws [Eq. (5.106)] are true, Bachelot uses the conservation laws, as follows:

$$\int_{\mathbf{R}^3} |\psi|^2 dx = \text{Const}, \quad \int_{\mathbf{R}^3} \tilde{\psi} \gamma_2 \psi dx = \text{Const}, \tag{5.108}$$

provided that these integrals do exist. This reservation must be demanded because we know that there are no bound states between an electric and a magnetic charge (Lochak, 1983, 1984), so that this result is not general.

Under the preceding restriction, we find from Eq. (5.108):

$$\int_{\mathbf{R}^3} \left| \Psi - e^{i\theta} \gamma_2 \psi^* \right|^2 dx = 2 \int_{\mathbf{R}^3} \left\{ |\Psi|^2 - \Re e^{-i\theta} \tilde{\psi} \gamma_2 \psi^* \right\} dx = \text{Const}. \tag{5.109}$$

Therefore, if at a given instant, Eq. (5.95), or, equivalently, Eq. (5.96) is realized, it also will be realized in the future. This is known as the *lemma of Bachelot,* and we know that it is true not only for Eq. (5.104), but also for Eq. (5.105).

If the preceding formulas are true, the condition to which Eqs. (5.100) and (5.101) were submitted through the Lagrange multipliers will be strongly weakened because instead of a constraint imposed at every instant, we have only an initial condition. Therefore, the Majorana magnetic states are simply particular solutions of the Dirac equation that have a magnetic interaction.

More precisely, these states are *monopole states of the Dirac equation* because it will be shown that Eq. (5.100) or (5.101) is actually chiral-invariant, despite the fact that Eq. (5.105) is not chiral-invariant. They represent a couple of monopoles and, in order to make them appear, it is sufficient to satisfy an initial condition, at least in certain cases.

Let us emphasize, once more, that by virtue of Eq. (5.107), this conclusion, which is true in the magnetic case, is not true in the electric case. So that, if we are able to satisfy the conditions [Eq. (5.95)], we shall obtain monopoles, but not electrons.

5.10 ANOTHER POSSIBLE EQUATION: THE GAUGE INVARIANCE PROBLEM

At this point, let us introduce the transformation $\Psi \to e^{ig\hbar c\gamma_5\Phi}\Psi$ (discussed in Chapter 4) in Eq. (5.100). Here, we find

$$\gamma_\mu\left[\partial_\mu\Psi - g\hbar c\gamma_5\left(B_\mu + i\partial_\mu\Phi\right)\right] e^{ig\hbar c\gamma_5\Phi}\Psi - m_0c/\hbar e^{i\theta}\gamma_2 e^{-ig\hbar c\gamma_5\Phi}\psi^* = 0. \tag{5.110}$$

And then, taking into account the anticommutation rules of γ matrices:

$$\gamma_\mu\left[\left(\partial_\mu\Psi - g\hbar c\gamma_5\left(B_\mu + i\partial_\mu\Phi\right)\right]\Psi - m_0c/\hbar e^{i\theta}\gamma_2\psi^* = 0. \tag{5.111}$$

We find the correct interaction term with the B_μ potentials, but with a phase factor Φ, the origin of this factor is the angle A. The chiral gauge invariance is not obvious, as could be expected, because this invariance in its general form appears only in the equations in which the chiral angle A does not appear, whereas in the present case, we started from Eq. (5.105), which is not chiral gauge-invariant. We have just imposed one of the conditions [i.e., Eq. (5.95)], which does not make the angle A disappear, but it is undetermined. That is, this angle appears in the equation, but its value can be eliminated if it is a polar angle around a nil rotation vector.

Finally, the preceding phase factor is eliminated by a choice of angle A because the gauge invariance is lost, and the phase θ may be eliminated as well because it plays no dynamical role. Thus, we can write, as a consequence of Eq. (5.95):

$$\Psi = \gamma_2\psi^* = \psi_c\xi = is_2\eta^*\eta = -is_2\xi^*. \tag{5.112}$$

And instead of Eqs. (5.100) and (5.101), we have (without θ)

$$\gamma_\mu\left(\partial_\mu - g\hbar c\gamma_5 B_\mu\right)\Psi - m_0c\hbar\gamma_2\psi^* = 0 \tag{5.113}$$

and [see Eq. (5.102)]

$$\begin{aligned}\left(\pi_0^+ + \pi^+\cdot s\right)\xi - im_0cs_2\xi^* &= 0\\ \left(\pi_0^- + \pi^-\cdot s\right)\eta + im_0cs_2\eta^* &= 0.\end{aligned} \tag{5.114}$$

5.11 GEOMETRICAL OPTIC APPROXIMATION

Until now, all seems well, but it would be desirable to test the qualities of the preceding equations for a well-known case, such as the interaction of a monopole with an electric charge, as it was done previously in the comparable case of the electron, replacing Eq. (5.14) with Eq. (5.114). This seems easy because of the apparent identity of both equations. Unfortunately, that is not so because the potentials hidden in these formulas are fundamentally different, so the magnetic case is far more complicated than the electric one. And this case is more difficult than the case of a linear, massless monopole (as discussed in Chapter 3), precisely because of the nonlinear mass term.

For these reasons, we shall be content with the classical approximation. Thus, we shall take Eq. (5.114) with π^+ and π^- defined in Eq. (5.102), with the following expressions:

$$\xi = a\exp(-iS/\hbar) + b\exp(iS/\hbar), \quad \eta = -is_2\xi^*. \tag{5.115}$$

A calculation analogous to the one in § 5.6 gives an equation of the Hamilton-Jacobi type:

$$\left[\left(\frac{1}{c}\frac{\partial S}{\partial t}\right)^2 - \left(\nabla S + \frac{g\mathbf{B}}{c}\right)^2 - m_0^2c^2\right]\left[\left(\frac{1}{c}\frac{\partial S}{\partial t}\right)^2 - \left(\nabla S - \frac{g\mathbf{B}}{c}\right)^2 - m_0^2c^2\right] = 4m_0^2g^2\mathbf{B}^2. \tag{5.116}$$

This is very different from Eq. (5.61) because of the difference in the potentials (see Eq. (1.26) in Chapter 1):

$$W = 0,\ B_x = \frac{e}{r}\frac{yz}{x^2+y^2},\ B_y = \frac{e}{r}\frac{-xz}{x^2+y^2},\ B_z = 0,\quad r = \sqrt{x^2+y^2+z^2}. \tag{5.117}$$

The electric coulomb field is written as

$$\mathbf{E} = \mathit{curl}\,\mathbf{B} = e\mathbf{r}/r^3. \tag{5.118}$$

Now, it must be recognized that Eq. (5.116) is not the equation of a classical magnetic monopole in the presence of an electric charge, which would be

$$\mathbf{E} = \mathit{curl}\,\mathbf{G} = e\mathbf{r}/r^3$$

and so

$$\mathit{either}:\ \left[(\partial S/\partial t)^2/c^2 - (\nabla S + g\mathbf{B}/c)^2 - m_0^2c^2\right] = 0 \tag{5.119}$$

$$\mathit{or}:\ \left[(\partial S/\partial t)^2/c^2 - (\nabla S - g\mathbf{B}/c)^2 - m_0^2c^2\right] = 0. \tag{5.120}$$

depending on the sign of the magnetic charge. The simultanuous presence of both brackets, $(\nabla S + g\,\mathbf{B}/c)$ and $(\nabla S - g\,\mathbf{B}/c)$, in Eq. (5.116) suggests that the equation contains a couple of monopoles of opposite signs, and one can observe that, far from the center of the electric charge, $B \to 0$; and that Eq. (5.116) splits into two components: Eqs. (5.119) and (5.120). Thus, we actually find, asymptotically, a couple of classical monopoles. This may be called the *zero approximation*. The first-order equations (nearer to the center) may be written as

$$\begin{aligned}\left[(\partial S/\partial t)^2/c^2 - \left(\vec{\nabla} S + g\,\mathbf{B}/c\right)^2 - m_0^2 c^2\right] &= 2m_0 g|\mathbf{G}| \\ \left[(\partial S/\partial t)^2/c^2 - \left(\vec{\nabla} S - g\,\mathbf{B}/c\right)^2 - m_0^2 c^2\right] &= -2m_0 g|\mathbf{G}|.\end{aligned} \tag{5.121}$$

These equations have additive terms with respect to Eqs. (5.119) and (5.120).

It is interesting to introduce, in one of these equations, these two quantities:

$$p = \nabla S = m\frac{d\mathbf{r}}{dt}, \quad \lambda = \frac{egc}{\varepsilon} \quad (\varepsilon = \text{energy}), \tag{5.122}$$

which results in the following equation:

$$d^2 r/dt^2 = -\lambda/r^3 \cdot dr/dt \times r - m_0 g|\mathbf{B}|. \tag{5.123}$$

Without the second term of the second member, this is the Poincaré equation [i.e., Eq. (1.2)] of the interaction between an electric and a magnetic charge in classical mechanics. Remember that we already obtained such an equation, at the geometrical optic limit of our equation of a massless monopole in Chapter 3.

Nevertheless, we cannot neglect this strange additional term that appears in Eq. (5.123), and which is the same term as in Eqs. (5.116), (5.121), and (5.122): a kind of reminder of the sin of introducing a linear mass term, which disappears far from the center of the charge but which calls for a certain caution concerning the "Majorana monopole."

It must be added that the importance of the Poincaré equation comes not only from the fame of its author, but from the fact that this equation is experimentally verified by the Birkeland effect (as discussed in Chapter 1). It is the equation of the motion of a beam of cathodic rays in the presence of a pole of a linear magnet—actually, a magnetic monopole. This is why we have attached great importance to the fact that the Poincaré equation is the classical limit of our equation of the magnetic monopole.

It is, thus, impossible to be indifferent to a violation of the Poincaré equation. Yet it is not a total invalidation of the Majorana monopole because the additive term is always the same, and it tends to zero in two cases:

1. If the proper mass tends to zero, *this monopole tends to our massless monopole*. But this case is not significant because *it is evident from* Eqs. (5.113) *and* (5.114) *that the nonlinear Majorana term tends to zero.*
2. *Far from the center of the charge*, which is due to the potentials and is not evident from Eqs. (5.113) and (5.114). This gives the Majorana monopole an asymptotic significance. However, it must be noted that, far from the center, the additive terms become negligible; but unfortunately, the potential terms become negligible too, so *the Majorana monopole tends to our massless monopole when it is no more a monopole.*

APPENDIX A

In this appendix, let us give a proof of Eq. (4.6). First, we know that by the very definition of $\Omega_l^m(-)$ *and* $\Omega_l^m(+)$, we get

$$\mathbf{J}^2\Omega_{l-1}^m(+) = j(j+1)\Omega_{l-1}^m(+), \quad j = l - \frac{1}{2} \text{ and} \tag{A.1}$$

$$\mathbf{J}_z\Omega_l^m(-) = u\Omega_l^m(-), \ \mathbf{J}_z\Omega_{l-1}^m(+) = u\Omega_{l-1}^m(+), \quad u = m - \frac{1}{2}, \tag{A.2}$$

where we have

$$\mathbf{J} = \mathbf{L} + \mathbf{s}, \quad \mathbf{L} = -i \ \mathbf{r} \times \nabla. \tag{A.3}$$

Now, we can easily verify that the operator

$$\mathbf{s.n} = \frac{1}{r}\mathbf{s.r} = \begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix} \tag{A.4}$$

commutes with $\mathbf{J}$:

$$[\mathbf{J}, \mathbf{s.n}] = 0. \tag{A.5}$$

Therefore, $\mathbf{s.n}$ transforms a subspace Ω that belongs to the subspace of eigenvalues of J^2 and J_z in an element of the same subspace. For instance, we have

$$\mathbf{s.n}\ \Omega_l^m(+) = A\Omega_l^m(+) + B\Omega_{l+1}^m(-), \tag{A.6}$$

where the constants A and B do not depend on m. We shall compute them for particular values of m and of the polar angles as follows:

$$m = l + 1, \quad \theta = \frac{\pi}{2}, \quad \phi = 0. \tag{A.7}$$

From Eq. (5.21), we have

$$\Omega_l^{l+1}(+) = \begin{bmatrix} Y_l^l\left(\frac{\pi}{2}, 0\right) \\ 0 \end{bmatrix}, \quad \Omega_{l+1}^{l+1}(-) = \begin{bmatrix} \left(\frac{l}{2l+3}\right)^{12} Y_{l+1}^l\left(\frac{\pi}{2}, 0\right) \\ -\left(\frac{2l+2}{2l+3}\right)^{12} Y_{l+1}^{l+1}\left(\frac{\pi}{2}, 0\right) \end{bmatrix}. \tag{A.8}$$

Now, from Eq. (5.22),

$$Y_{l+1}^l\left(\frac{\pi}{2}, 0\right), \quad \left(\frac{2l+2}{2l+3}\right)^{12} Y_{l+1}^{l+1}\left(\frac{\pi}{2}, 0\right) = -Y_l^l\left(\frac{\pi}{2}, 0\right), \tag{A.9}$$

and Eq. (A.4) gives

$$\mathbf{s.n}\ \left(\frac{\pi}{2}, 0\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{A.10}$$

Finally, it is sufficient to introduce Eqs. (A.8), (A.9), and (A.10) into Eq. (A.7) to find

$$A = 0, \quad B = 1, \tag{A.11}$$

which proves the first relation [Eq. (5.23)]. The second relation is evident because

$$(\mathbf{s.n})^2 = I. \tag{A.12}$$

Thus, we have

$$Y_l^{m*}(\theta, \phi) = (-1)^m Y_l^{-m}. \tag{A.13}$$

APPENDIX B

To prove Eq. (5.29), remember that in Eq. (A.3), **L** and **s** commute, so from Eqs. (5.21) and (A.1), we have

$$\mathbf{J}^2\Omega_l^m(\pm) = j(j+1)\Omega_l^m(\pm), \quad \mathbf{L}^2\Omega_l^m(\pm) = l(l+1)\Omega_l^m(\pm)$$
$$\mathbf{S}^2\Omega_l^m(\pm) = s(s+1)\Omega_l^m(\pm) = \frac{3}{4}\Omega_l^m(\pm). \tag{B.1}$$

Thus, applying Eq. (A.3), we get

$$\begin{aligned}(\mathbf{L}+\mathbf{S})^2\Omega_l^m(\pm) &= \left(\mathbf{L}^2+\mathbf{S}^2+2\mathbf{L}.\mathbf{S}\right)\Omega_l^m(\pm)\\ &= \left(\mathbf{L}^2+\mathbf{S}^2+\mathbf{L}.\mathbf{S}\right)\Omega_l^m(\pm)\end{aligned} \tag{B.2}$$

so that Eq. (B.1) gives $j(j+1)\Omega_l^m(\pm) = \left[l(l+1)+\frac{3}{4}+\mathbf{L}.\mathbf{s}\right]\Omega_l^m(\pm)$ and Eq. (5.29).

CHAPTER 6

A New Electromagnetism with Four Fundamental Photons: Electric, Magnetic, with Spin 1 and Spin 0

6.1 THEORY OF LIGHT

6.1.1 Theory of Light and Wave Mechanics: A Historical Summary

This chapter presents an introduction to a new theory of light and gravitation (the last at a linear approximation) that generalizes, owing to the idea of the magnetic monopole, the de Broglie theory of light and gravitation based on his theory of spin particles. The idea of leptonic monopole—and its consequences—are the new concepts added to de Broglie's theory. On the contrary, other ideas that appear in the new theory, including the "magnetic photon," were implicitly present (in a hidden form) in the de Broglie theory of spin particles; but curiously, they remain unexploited (or even noticed) until recent years. This is the reason for the following short historical summary.

The de Broglie theory of spin particles started from his work on the theory of light that began as a dynamic theory of the Einstein photon. At that time (1922), the wave mechanics did not yet exist: it appeared a little later, precisely from this dynamic theory of Einstein's "light quanta" (de Broglie, 1922).

De Broglie initially tried a test of the photon hypothesis, going as far as possible with the radiation theory, in a purely corpuscular way, in the spirit of Newton, but introducing relativistic mechanics, kinetic theory, and thermodynamics; nevertheless, they did not use electromagnetism because de Broglie *aimed to find where and in what form the waves become necessary.*

He considered Einstein's "light quanta," which were not yet called *photons,* to be true particles (as he put it, "atoms of light") with a *small proper mass,* obeying the laws of relativistic mechanics. Starting from a purely corpuscular point of view, he got several results previously considered as the consequences of electromagnetism:

- For instance, if $E = mc^2 = 1/\sqrt{1 - \mathrm{v}^2/c^2}$ = *total energy*, the relativistic form of the momentum B is $B = mc = E/c$, from which de Broglie obtained the correct relation $p = \rho/3$ between the pressure and energy density of black radiation (de Broglie, 1922), first proved by Boltzmann and later ascribed to Maxwell's theory.[11]
- Applying relativity, de Broglie gave the correct mean energy 3 kT for the photon instead of the half-value $(3/2)kT$ of the classical theory of gas. This energy was usually considered as the sum of electric and magnetic energies, whereas it is a simple consequence of relativistic kinematics.
- Finally, de Broglie obtained the formula of the Doppler effect from the relativistic addition of velocities and Planck's law of quanta.

After these results, de Broglie realized that his ideas were not restricted to light and photons, but rather could be said about every particle. Therefore, he attached a frequency to each material particle via the expression $mc^2 = h\nu$. This brought him, if not yet to the wave, at least to a frequency that he ascribed to an "internal clock" of the particle, which was not far from Newton's conceptions. But he rapidly understood that such an interpretation is not relativistically invariant because if ν is an internal frequency of a particle, it is submitted to the slowing of the clocks, while m will increase with the velocity. The de Broglie "illuminating idea" (in his own words) was that, on the contrary, the frequency of a wave would have the same variance as m so that the expression $mc^2 = h\nu$ becomes relativistically invariant and defines univocally ν from m. This was the start of wave mechanics.

[11] It is curious to note that Planck found twice this result, due to the omission of relativity (which is absolutely astonishing coming from Max Planck). So, he wrote $E = (1/2)m\mathrm{v}^2 \Rightarrow B = m\mathrm{v} = 2W/\mathrm{v} \Rightarrow$ $\mathrm{p} = 2\rho/3$, with an erroneous factor of 2, considered by the opponents to Einstein as an argument against the photon hypothesis (*de Broglie*, 1922) …

It must be stressed that de Broglie considered from the very beginning that the photon had a mass: namely, a mass far smaller than the one of an electron, but it was a "true" mass that includes the photon in a description of all the particles of the universe. Nevertheless, such a theory of light could not be developed with the Schrödinger or Klein-Gordon equation because the first is nonrelativistic and the waves of both equations are not polarized.

The situation became different with the appearance of the Dirac equation for which de Broglie was immediately enthusiastic because he saw in it a possible beginning for a theory of light (de Broglie, 1932a–c): the equation was relativistic, with a four-component wave function (and, therefore, a polarization) and a spin: the axial vector that he had predicted for light[12]; and a second-rank tensor, $M_{\mu\nu} = \overline{\psi}\gamma_\mu\gamma_\nu\psi$, which is antisymmetric as the electromagnetic tensor, despite the fact that it was not a wave.

Nevertheless, the elements of the Dirac equation could not be directly applied to a photon: the wave does not have the variance either of a vector or of an antisymmetric tensor [such a tensor $(\overline{\psi}\gamma_\mu\gamma_\nu\psi)$ is present in the theory, but it is not the wave]; the spin rotates twice as slow and the particle is a fermion, not a boson, as was already well known. Nevertheless, the way was not obstructed as it had been because the different elements did exist, but in a distorted form.

After some initial attempts (de Broglie, 1932a, b), de Broglie realized that a photon cannot be an elementary particle, but the fusion of a pair: perhaps of a spin-1/2 corpuscle and its "anticorpuscle" (this word appearing here for the first time), both obeying a Dirac equation (de Broglie, 1932b).

The creation and annihilation of pairs suggested that a photon could result from the "fusion" of an electron-positron pair linked by an electrostatic force. The smallness of the photon mass could be a consequence of a defect in relativistic mass. But the introduction of an electrostatic force is a source of confusion because a theory of photons is a theory of electromagnetism, so the electrostatic force must be a *consequence* of the theory, not an a priori hypothesis.

[12] De Broglie (1922) wrote: "A more complete theory of quanta of light must introduce a polarization in such a way that: to each atom of light would be linked an internal state of right or left polarization represented by an axial vector with the same direction as the propagation velocity." It was shown later that when the velocity of a particle tends to the velocity of light, the space components of the vector spin lies along the velocity.

So, recognizing that the choice of conjugated particles was impossible, de Broglie supposed that the photon is a neutrino-antineutrino pair or, more generally, the center of mass of a couple of Dirac particles. He published the equation in 1932 (de Broglie 1932b, c) and he developed the theory during many years.

6.1.2 De Broglie's Method of Fusion

First, let us take as an example a pair of identical, ordinary particles of mass m, obeying the Schrödinger equation, with respective coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2). Their center of mass is

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2}. \tag{6.1}$$

The Schrödinger equation of the center of mass is definite, using the coordinates in Eq. (6.1):

$$-i\hbar \frac{\partial \phi}{\partial t} = \frac{1}{2M} \Delta \phi \quad (M = 2m). \tag{6.2}$$

But such a procedure cannot be extended to a pair of Dirac particles because there is no quantum (or even a classical) relativistic theory of systems of particles. Therefore, de Broglie suggested a formal way that is easier to generalize. He associated the particles with two different waves, ψ and φ, without making any distinction between their coordinates. So we have the following equations with the same coordinates x_k:

$$-i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} \Delta \psi; \quad -i\hbar \frac{\partial \varphi}{\partial t} = \frac{1}{2m} \Delta \varphi \tag{6.3}$$

Now, the fusion conditions, expressing the equality of moment and energy in the case of plane waves, are

$$\frac{\partial \psi}{\partial t} \varphi = \psi \frac{\partial \varphi}{\partial t} = \frac{1}{2} \frac{\partial (\psi \varphi)}{\partial t}; \quad \frac{\partial^2 \psi}{\partial x_k^2} \varphi = \frac{\partial \psi}{\partial x_k} \frac{\partial \varphi}{\partial x_k} = \psi \frac{\partial^2 \varphi}{\partial x_k^2} = \frac{1}{4} \frac{\partial^2 (\psi \varphi)}{\partial x_k^2}. \tag{6.4}$$

Multiplying the first equation in Eq. (6.3) by φ and the second by ψ, we find for $\phi = (\varphi \psi)$ Eq. (6.2) again. Then de Broglie applied the same conditions to all the waves without restriction to the plane waves, and he applied it to the relativistic case: it is called the *fusion postulate*.

6.1.3 De Broglie's Equations of Photons

Consider the Dirac equations of two particles of mass $\frac{\mu_0}{2}$:

$$\begin{aligned}\frac{1}{c}\frac{\partial\psi}{\partial t} &= \alpha_k\frac{\partial\psi}{\partial x_k} + i\frac{\mu_0 c}{2\hbar}\alpha_4\psi \\ \frac{1}{c}\frac{\partial\varphi}{\partial t} &= \alpha_k\frac{\partial\varphi}{\partial x_k} + i\frac{\mu_0 c}{2\hbar}\alpha_4\varphi\ ,\end{aligned} \tag{6.5}$$

where $\{\alpha_k,\ \alpha_4\}$ are the Dirac matrices[13]:

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix};\ \ \alpha_k = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix};\ \ (\sigma_k = Pauli\ \ matrices). \tag{6.6}$$

In analogy with Eq. (6.4), de Broglie put the *fusion conditions* on the Dirac wave-components as follows:

$$\frac{\partial\psi_n}{\partial t}\varphi_m = \psi_n\frac{\partial\varphi_m}{\partial t} = \frac{1}{2}\frac{\partial(\psi_n\varphi_m)}{\partial t};\quad \frac{\partial\psi_n}{\partial x_k}\varphi_m = \psi_n\frac{\partial\varphi_m}{\partial x_k} = \frac{1}{2}\frac{\partial(\psi_n\varphi_m)}{\partial x_k}. \tag{6.7}$$

So he found for $\phi = \{\phi_{nm} = \psi_n\varphi_m\}$ a new equation, which he extended by postulate to all the ϕ functions even if their form is not $\phi_{nm} = \psi_n\varphi_m$:

$$\begin{aligned}\frac{1}{c}\frac{\partial\phi}{\partial t} &= a_k\frac{\partial\phi}{\partial x_k} + i\frac{\mu_0 c}{\hbar}a_4\phi \\ \frac{1}{c}\frac{\partial\phi}{\partial t} &= b_k\frac{\partial\phi}{\partial x_k} + i\frac{\mu_0 c}{\hbar}b_4\phi.\end{aligned} \tag{6.8}$$

The matrices a and b are defined as

$$\begin{aligned}&a_r = \alpha_r \times I,\ \ (a_r)_{ik,lm} = (\alpha_r)_{il}\delta_{km} \\ &b_r = I \times \alpha_r,\ \ (b_r)_{ik,lm} = (-1)^{r+1}(\alpha_r)_{km}\delta_{il}\quad (r = 1,2,3,4).\end{aligned} \tag{6.9}$$

They separately verify the relations of the Dirac matrices, where a and b commute:

$$a_r a_s + a_s a_r = 2\delta_{rs};\ \ b_r b_s + b_s b_r = 2\delta_{rs};\ \ a_r b_s - b_s a_r = 0. \tag{6.10}$$

With this finding, it is easy to prove that the components of ϕ obey the Klein-Gordon equation.

Eq. (6.8) with the definitions (6.9) are the *de Broglie photon equations* and we shall see that they include the Maxwell equations.

[13] For the beginning of the theory, we keep the old notations that de Broglie used.

First, however, we must examine some other representations of the photon equations:

The Quasi-Maxwellian Form

First, it must be noted that there are too many equations in Eq. (6.8): 32 equations for only 16 components of the wave ϕ. There is a problem of compatibility. To solve the problem, de Broglie added and subtracted the two systems in Eq. (6.8):

$$(A)\ \ \frac{1}{c}\frac{\partial\phi}{\partial t}=\frac{a_k+b_k}{2}\frac{\partial\phi}{\partial x_k}+i\frac{\mu_0 c}{\hbar}\frac{a_4+b_4}{2};\quad (B)\ \ 0=\frac{a_k-b_k}{2}\frac{\partial\phi}{\partial x_k}+i\frac{\mu_0 c}{\hbar}\frac{a_4-b_4}{2}\phi. \tag{6.11}$$

Furthermore, it will be shown that Eq. (6.8) exactly contains the Maxwell equations (up to mass terms), but Eq. (6.11) is already an outline of these equations because this system is divided into a group (A) of "evolution equations" that resembles the Maxwell equations in $\partial\mathbf{E}/\partial\mathrm{t}$ and $\partial\mathbf{H}/\partial\mathrm{t}$, and a group ($B$) of "condition equations," of the same kind as $div\mathbf{E}=0$ and $div\mathbf{H}=0$. In de Broglie (1934b), it gave only the group (A), but it is easy to prove, in analogy with the Maxwell equations, the following:

- Owing to Eq. (6.10), (B) is a consequence of (A).
- Actually, (B) is only satisfied by the solutions of (A) whose Fourier expansion does not contain a zero frequency. But the zero frequencies are automatically absent from the solutions of (A) if $\mu_0 \neq 0$.
- Therefore, *iff* $\mu_0 \neq 0$, the condition (B) is a consequence of the evolution equations (A).

Canonical Form

Eq. (6.8) can be transformed in another way:

$$\begin{aligned}(C)\ \ \frac{1}{c}\frac{a_4+b_4}{2}\frac{\partial\phi}{\partial t}&=\frac{b_4a_k+a_4b_k}{2}\frac{\partial\phi}{\partial x_k}+i\frac{\mu_0 c}{\hbar}a_4b_4\phi\\ (D)\ \ \frac{1}{c}\frac{a_4-b_4}{2}\frac{\partial\phi}{\partial t}&=\frac{b_4a_k-a_4b_k}{2}\frac{\partial\phi}{\partial x_k}.\end{aligned} \tag{6.12}$$

This new system is at the basis of the Lagrangian derivation of the theory and of its tensorial form, and it was used by de Broglie to quantize the photon field and to describe the photon-electron interaction (de Broglie, 1940–1942). Just as in Eq. (6.11), (D) is a consequence of (C) *if* $\mu_0 \neq 0$, which is proved by applying to (C) the operator: $\frac{1}{c}\frac{a_4-b_4}{2}\frac{\partial}{\partial t}$, taking into account Eq. (6.10). It is noteworthy that the strongest arguments in favor of a massive photon are not the answers to particular experimental objections, but the arguments imposed by the fusion theory, which are linked to the very structure of the theory.

6.1.4 Introduction of a Square-Matrix Wave Function

Now, let us return to the initial system [Eq. (6.8)], but in terms of relativistic coordinates $x_k = (x, y, z),\ \ x_4 = ict$ with γ matrices $(\mu, \nu = 1, 2, 3, 4)$:

$$\begin{aligned}&\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu};\ \ \mu, \nu = 1, 2, 3, 4;\ \ \gamma_k = i\alpha_4\alpha_k;\\&\gamma_4 = \alpha_4;\ \ \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4.\end{aligned} \tag{6.13}$$

Multiplying Eq. (6.8) by $i\alpha_4$, we find that owing to Eq. (6.9), the following new system, which is not written in terms of a 16-line column wave function ϕ but in terms of a 4 ×4 square matrix wave function ψ:

$$\begin{aligned}&\partial_\mu\gamma_\mu\Psi - \frac{\mu_0 c}{\hbar}\Psi = 0\\&\partial_\mu\Psi\tilde{\gamma}_\mu - \frac{\mu_0 c}{\hbar}\Psi = 0\end{aligned}\quad \big(\mu, \nu = 1, 2, 3, 4;\ \ \tilde{\gamma}_\mu = \gamma_\mu\ \ \mathit{transp.}\big). \tag{6.14}$$

The transposed matrices $\tilde{\gamma}$ are easily eliminated because, if two sets of Dirac matrices γ_μ and $\tilde{\gamma}_\mu$, verify the relations [Eq. (6.13)], there are two (and only two) nonsingular matrices, Λ **and** Γ, such that

$$\tilde{\gamma}_\mu = \Lambda\gamma_\mu\Lambda^{-1};\ \ \tilde{\gamma}_\mu = -\Gamma\gamma_\mu\Gamma^{-1};\ \ \Lambda = \Gamma\gamma_5;\ \ \mu = 1, 2, 3, 4. \tag{6.15}$$

γ_5 is given in Eq. (6.13), and Eq. (6.15) is true for $\tilde{\gamma}_\mu$ transposed from $\gamma_{\mu;}$ A solution is as follows:

$$\Gamma = -i\gamma_2\gamma_4;\ \ \Lambda = \Gamma\gamma_5 = -i\gamma_3\gamma_1. \tag{6.16}$$

The Λ case in Eq. (6.15) was given in Pauli (1936), and the Γ case was given by de Broglie to eliminate $\tilde{\gamma}_\mu$ in Eq. (6.16). Indeed, introducing Γ into Eq. (6.14), we find the system given by Tonnelat, de Broglie, and Pétiau (Tonnelat, 1938; de Broglie, 1940−1942):

$$\begin{aligned}&\partial_\mu\gamma_\mu(\psi\Gamma) - \frac{\mu_0 c}{\hbar}(\psi\Gamma) = 0\\&\partial_\mu(\psi\Gamma)\gamma_\mu + \frac{\mu_0 c}{\hbar}(\psi\Gamma) = 0.\end{aligned} \tag{6.17}$$

The equations obtained by substituting Λ to Γ [Eq. (6.15)] were given recently (Lochak, 2014):

$$\partial_\mu \gamma_\mu (\psi \Lambda) - \frac{\mu_0 c}{\hbar}(\psi \Lambda) = 0$$
$$\partial_\mu (\psi \Lambda)\gamma_\mu - \frac{\mu_0 c}{\hbar}(\psi \Lambda) = 0. \quad (6.18)$$

The apparently small formal difference (a minus sign) between the two systems [Eqs. (6.17) and (6.18)] entails a great physical difference because the solutions of these equations exchange between themselves by a mutiplication by γ_5: they are dual in space-time, and we shall prove that it signifies the exchange between electric and magnetic charges.

So, the substitution of Λ to Γ in the representation by square matrices of the initial de Broglie's equations [Eq. (6.5)] gives two kinds of photons:

- Electric and magnetic photons
- The electromagnetic formulas of the photon equations

The fundamental electromagnetic formulas were given by de Broglie in his first papers, starting from Eq. (6.8) (de Broglie 1934a, b, 1936). For the sake of simplicity, we start from Eqs. (6.17) and (6.18), applying a procedure suggested by M.A. Tonnelat and then used by de Broglie (1934b).

Let us expand a 4 ×4 matrix $\boldsymbol{\Theta}$ on the Clifford algebra in $\mathbb{R}^{+---}$:

$$\Lambda\Psi = \boldsymbol{\Theta} = I\varphi_0 + \gamma_\mu \varphi_\mu + \gamma_{[\mu\nu]}\varphi_{[\mu\nu]} + \gamma_\mu \gamma_5 \varphi_{\mu 5} + \gamma_5 \varphi_5 \,, \quad (6.19)$$

where φ_0 is a scalar, φ_μ a polar vector, $\varphi_{[\mu\nu]}$ an antisymmetric tensor of rank 2, $\varphi_{\mu 5}$ an axial vector (the dual of an antisymmetric tensor of rank 3) and φ_5 a pseudoscalar (the dual of an antisymmetric tensor of rank 4). These expressions correspond in $\mathbb{R}^3$ to a scalar I_1; the Lorentz potentials **A**, V (linked to the electric charges); the electromagnetic fields **H**, **E**; the pseudo potentials **B**, W (linked to magnetic charges); and a pseudo-scalar I_2[14]:

$$\mathbf{H} = Kk_0\left(\varphi_{[23]}, \varphi_{[31]}, \varphi_{[12]}\right); \;\; \mathbf{E} = Kk_0\left(i\varphi_{[14]},\; i\varphi_{[24]},\;\; i\varphi_{[34]}\right)$$
$$\mathbf{A} = K(\varphi_1,\; \varphi_2,\; \varphi_3); \;\; iV = K\varphi_4$$
$$-\, i\mathbf{B} = K\left(\varphi_{[15]},\; \varphi_{[25]},\; \varphi_{[35]}\right); \;\; W = K\varphi_{[45]} \quad (6.20)$$
$$I_1 = K\varphi_0; \;\; iI_2 = K\varphi_5 \;\; \left(k_0 = \frac{\mu_0 c}{\hbar}; \;\; K = \frac{\hbar}{2\sqrt{\mu_0}}\right).$$

[14] Remember that **B** is not an induction. The Lorentz polar quadripotential (V, **A**) remains linked to the electric Einstein photon, while the pseudoquadripotential (W,**B**) is linked to the magnetic photon.

Now, if we develop Eqs. (6.17) and (6.18) owing to Eqs. (6.19) and (6.20), we find two sets of equations, discussed in the next sections.

6.1.4 The Equations of the "Electric Photon" (Γ Matrix).

The expansion of the matrix wave-function $\boldsymbol{\Psi} = \psi\,\boldsymbol{\Gamma}$ according to Eq. (6.19) splits Eq. (6.17) into two systems (de Broglie 1940–1942, 1943), that we now refer to as the *electric photon* because the vector potential **A** appears in (6.21):

$$(\mathrm{M})\quad \left(\begin{array}{l} -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t} = curl\mathbf{E};\quad \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t} = curl\,\mathbf{H} + k_0^2\mathbf{A} \\ div\mathbf{H} = 0;\quad div\mathbf{E} = -k_0^2 V \\ \mathbf{H} = curl\,\mathbf{A};\quad \mathbf{E} = -grad V - \dfrac{1}{c}\dfrac{\partial \mathbf{A}}{\partial t};\quad \dfrac{1}{c}\dfrac{\partial V}{\partial t} + div\mathbf{A} = 0 \end{array}\right) \tag{6.21}$$

$$(\mathrm{NM})\quad \left(\begin{array}{l} -\dfrac{1}{c}\dfrac{\partial I_2}{\partial t} = k_0 W;\quad grad\ \ I_2 = k_0\mathbf{B};\quad \dfrac{1}{c}\dfrac{\partial W}{\partial t} + div\ \mathbf{B} = k_0 I_2 \\ curl\ \mathbf{B} = 0;\quad grad W + \dfrac{1}{c}\dfrac{\partial \mathbf{B}}{\partial t} = 0;\quad \{(k_0 I_1 = 0;\ \ k_0 \neq 0) \Rightarrow I_1 = 0\} \end{array}\right). \tag{6.22}$$

Actually, de Broglie fixed his attention essentially on the first system of equations [Eq. (6.21)], which he denoted as (M) ("Maxwellian"), for obvious reasons, and he considered it as the equations of the photon (M: spin 1). This was the great victory of his theory: the deduction of Maxwell's equations from Dirac's equation.

Curiously, de Broglie was rather puzzled by the second system spin 0, that he named negatively: **NM** ("non-Maxwellian"), without giving any clear interpretation. He thought at first of a meson but then abandoned the idea. Here, we shall adopt the following very simple interpretation.

It is natural to find two systems of equations because the fundamental equations [Eq. (6.8)] are not the equations of a particle of spin 1, but of a particle of maximum spin 1: a combination of two particles of spin ½, as de Broglie underlined it. For this reason, just as for a diatomic molecule, we find two states described by two systems of equations: an orthostate of spin 1 = ½ + ½ (parallel spins) and a parastate of spin 0 = ½ − ½ (opposite spins). We shall adopt this interpretation.

Both states have equal rights with respect to the symmetry laws because both are linked by a symmetry of form and both have a physical sense, despite the fact that one of them (the orthostate: spin 1) is related to a far more celebrated case: the Maxwell equations, while the orthostate is related to the smaller Aharonov-Bohm effect, as will be shown later in this chapter.

Thus, we have two photons—more precisely, two spin states: 1 and 0, of a photon described by the systems [Eqs. (6.21) and (6.22)]. And it is not a general photon, but only an electric photon; this is because we shall find another one: a magnetic photon. For the moment, we have just the electric photon with two photon states: a spin 1 state (M), "Maxwellian," and a spin 0 state (NM), "non-Maxwellian."

The (M) equations are Maxwell's equations, but with two differences:

1. The first difference is the presence of the *mass terms,* which introduces a link between fields and potentials, the latter becoming physical quantities and losing their gauge invariance.
2. The second difference is the automatic definition of fields, through the Lorentz potentials, with the following Lorentz gauge condition:

$$\mathbf{H} = curl\ \mathbf{A};\quad \mathbf{E} = -\mathbf{grad}\,V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t};\quad \frac{1}{c}\frac{\partial V}{\partial t} + div\mathbf{A} = 0. \tag{6.23}$$

These relations are not arbitrarily added to the field equations, as they were in the classical theory: they appear automatically and they are themselves field equations, as a consequence of the massive photon. Of course, they were already present in a hidden form in Eqs. (6.8), (6.11), (6.12), and (6.17).

A consequence of Eq. (6.21). is that the fields and potentials do not obey the d'Alembert wave equation, but rather the Klein-Gordon equation:

$$\Box F + k_0^2 F = 0;\quad (F = \mathbf{E},\ \mathbf{H},\ \mathbf{A},\ V,\ \mathbf{B},\ W,\ I_1,\ I_2). \tag{6.24}$$

The electrostatic solution is not the Coulomb potential $\frac{1}{r}$, but the Yukawa potential $V = e^{-r/k_0}/r$, which remains a long-range potential because of the smallness of the Compton wave number $k_0 = \mu_0 c/\hbar$.

The (NM) equations were previously considered by de Broglie (as was already said) as describing an independent spin 0 meson with a far greater μ_0 mass than the mass of the photon. This fact is astonishing, considering that the equations (M) and (NM) came from the decomposition of the same system of equations, so that both rest masses are obliged to be equal. Of course, we shall abandon this idea, which was actually later abandoned by de Broglie himself. Our interpretation will be based, on the contrary, on the link between the two systems (M) and (NM), admitted as a fact.

The system (NM) describes a chiral particle because I_1 is a true invariant, but $I_1 = 0$; actually, the particle is definite by the second invariant I_2 which is a pseudoinvariant, dual of an antisymmetric tensor in $\mathbb{R}^{+---}$ (with $I_2 \neq 0$), and by the pseudo-quadrivector (**B**,W) in $\mathbb{R}^{+---}$.

It must be noted that de Broglie remarked (de Broglie, 1940–1942) that the situation could be interpreted in another way, defining a second electromagnetic field (which he called an "anti-field") which *equals zero* by virtue of Eq. (6.22), as follows:

$$\mathbf{H}' = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + grad\, W; \quad \mathbf{E}' = curl\, \mathbf{B}. \tag{6.25}$$

We shall follow the second interpretation here, on the basis of a symmetry between electricity and magnetism developed in our papers concerning the photon (Lochak 20) and the magnetic monopole (Lochak, 1992, 2000)[15]. We consider the systems [Eqs. (6.21)–(6.22)] as simply describing, with equal weight the orthohydrogene state (spin 1) and the parastate (spin 0) of an **electric photon**, for the following reasons:

1. In the system (M), we have an electromagnetic field $(\mathbf{E}, \mathbf{H})$ and a *polar* 4-potential $(V, \mathbf{A})$, related to $(\mathbf{E}, \mathbf{H})$ by the Lorentz formulas [Eq. (6.23)]. These fields and potentials enter into the dynamics of an *electric* charge. Because $k_0 \neq 0$, we have in general $div\mathbf{E} \neq 0$, so that the electric field **E** is not transversal, contrary to the magnetic field **H**: and **E** has a small longitudinal component, of the order of k_0.
2. In the (NM) equations, we have a *pseudo*-invariant I_2 and an *axial* 4-potential $(\mathbf{B}, W)$, to which may be added the invariant I_1 and the anti-field $\{\mathbf{E}', \mathbf{H}'\}$, defined in Eq. (6.25), and which will be related to magnetism. But here, $I_1 = \mathbf{E}' = \mathbf{H}' = 0$, which confirms the electric character of the (NM) photon by the annihilation of magnetic quantities.

The difference between the de Broglie interpretation and mine is that now (NM) is no longer separated from the spin 1 (state M): it is the spin 0 state of the same photon. The electric photon is the whole system [Eqs. (6.21)–(6.22)] with two values of spin.

6.1.5 The Equations of the Magnetic Photon (Λ Matrix).

This second photon is given by Eq. (6.18) with $\Lambda = \Gamma\gamma_5$ [Eq. (6.15)] instead of Γ in [Eq. (6.17)]. The primed new field components are the

[15] The de Broglie definition [Eq. (6.25)] of $\mathbf{H}'$ and $\mathbf{E}'$, in terms of a *pseudo*-quadripotential $(\mathbf{B}, W)$, was later rediscovered by Cabibbo and Ferrari (1962).

dual of the preceding ones, which means that the matrix γ_5 exchanges electricity and magnetism (Lochak, 1992, 2000):

$$(\mathrm{M})\quad \left(\begin{array}{l} -\dfrac{1}{c}\dfrac{\partial \mathbf{H}'}{\partial t} = curl\,\mathbf{E}' + k_0^2\mathbf{B}'; \quad \dfrac{1}{c}\dfrac{\partial \mathbf{E}'}{\partial t} = curl\,\mathbf{H}' \\ div\mathbf{H}' = k_0^2 W'; \quad div\mathbf{E}' = 0 \\ \mathbf{H}' = grad\,W' + \dfrac{1}{c}\dfrac{\partial \mathbf{B}'}{\partial t}; \quad \mathbf{E}' = curl\,\mathbf{B}'; \quad \dfrac{1}{c}\dfrac{\partial W'}{\partial t} + div\mathbf{B}' = 0 \end{array} \right) \tag{6.26}$$

$$(\mathrm{NM})\quad \left(\begin{array}{l} -\dfrac{1}{c}\dfrac{\partial I_1}{\partial t} = k_0 V'; \quad gradI_1 = k_0\mathbf{A}'; \quad \dfrac{1}{c}\dfrac{\partial V'}{\partial t} + div\mathbf{A}' = k_0 I_1 \\ curl\mathbf{A}' = 0; \quad grad\,V' + \dfrac{1}{c}\dfrac{\partial \mathbf{A}'}{\partial t} = 0; \quad \{(k_0 I_2 = 0; \ k_0 \neq 0) \Rightarrow I_2 = 0\} \end{array} \right) \tag{6.27}$$

The new photon is associated, as before, with a couple of fields. But the situation is inverted in the following ways:

1. The anti-field $(\mathbf{E}', \mathbf{H}')$ and the *axial* 4-potential $(W', \mathbf{B}')$ satisfy the Maxwell-type (M) system [Eq. (6.26)]. The definition [Eq. (6.25)] of the anti-fields now appears in Eq. (6.26) automatically (and not by an a priori definition), as one of the field equations. Now $(\mathbf{E}', \mathbf{H}')$ are not equal to zero. The fields $(\mathbf{E}', \mathbf{H}')$ are exactly those that enter into the dynamics of a magnetic charge: a monopole (Lochak 1985, 1995b and Chapters 2, 3 of this book).

   Besides, symmetrically to the electric case, we now have $div\mathbf{H}' \neq 0$, so that, in a plane wave, the *magnetic* field $\mathbf{H}'$ (instead of the electric one $\mathbf{E}'$) has a small longitudinal component of the order of k_0, while $\mathbf{E}'$ is transversal. We have a *magnetic photon*.
2. Now, the *polar* potentials $(V', \mathbf{A}')$ dual from $(W', \mathbf{S}')$ appear in the (NM) system (i.e., in the *spin 0 state*). The invariant I'_1 and the pseudoinvariant I'_2 invert their roles: we have now $I'_1 \neq 0$ and $I'_2 = 0$. The electromagnetic field $(\mathbf{E}', \mathbf{H}')$ defined by the Lorentz formulas [Eq. (6.23)] gives $(\mathbf{E}' \neq 0, \ \mathbf{H}' \neq 0)$ in the Maxwellian formulas (M) and $(\mathbf{E}' = \mathbf{H}' = 0)$ in the non-Maxwellian formulas (NM), as opposed to what we had in the electric case.

   It is a remarkable fact that de Broglie's *fusion* of two Dirac equations not only gives the classical Maxwell equations (as was proved by de Broglie), but also defines *two classes* of photons, corresponding respectively to electric or magnetic charges. The algebraic symmetry excludes any other possibility.

The symmetry between the two electromagnetic fields is all the more interesting in that such a symmetry already appears in the Dirac equation itself, in the form of *two minimal interactions* corresponding to electric and magnetic charges, associated with the two kinds of fields (Chapter 2 of this book and Lochak, 1995b). Symmetries of Dirac's and de Broglie's equations are thus mutually reinforced. Now we must address other issues, to wit:

- We have two kinds of photons: the electric and the magnetic photon.

But is their physical difference given by the difference between the two pairs of equations: Eqs. (2.10)−(2.11) and Eqs. (6.17)−(6.18) or the Dirac gauge and equation and the chiral gauge and equation? Yes, because it is the difference between the motion of an electron or a monopole in an electrodynamic field. For instance, in a linear electric field, the electron is linearly accelerated, while the monopole rotates around the field, and the reverse is true for a linear magnetic field.

- Actually, there are not only two but four kinds of photons because they can have a spin 1 or a spin 0.

The preceding answer is only related to spin 1. We must now answer a new question: are the spin 0 photons already known? The answer is yes and there is a wellknown example.

6.1.6 The Aharonov−Bohm Effect

Consider the equations of (NM) potentials: Eqs. (6.22) and (6.27):

Spin 0 electric photon: $-\frac{1}{c}\frac{\partial I_2}{\partial t} = k_0 W$; $grad\, I_2 = k_0\mathbf{B}$; $\frac{1}{c}\frac{\partial W}{\partial t} + div\,\mathbf{B} = k_0 I_2$ and the associated equations in Eq. (6.22).

Spin 0 magnetic photon: $-\frac{1}{c}\frac{\partial I_1}{\partial t} = k_0 V'$; $grad\, I_1 = k_0\mathbf{A}'$; $\frac{1}{c}\frac{\partial V'}{\partial t} + div\mathbf{A}' = k_0 I_1$ and the associated equations in Eq. (6.27).

We must remember that the spin 1 electric photon is associated with a magnetic spin 0 photon by the pseudo − invariant I_2, while the spin 1 magnetic photon is associated with an electric spin 0 photon by the true invariant I_1. The preceding relations immediately imply that the spin 0 potentials are the gradients of relativistic invariants, which verifiy the Klein-Gordon equation:

$$\partial_\mu I_1 = k_0 A_\mu; \quad \Box I_1 + k_0^2 I_1 = 0; \quad \partial_\mu I_2 = k_0 B_\mu; \quad \Box I_2 + k_0^2 I_2 = 0. \tag{6.28}$$

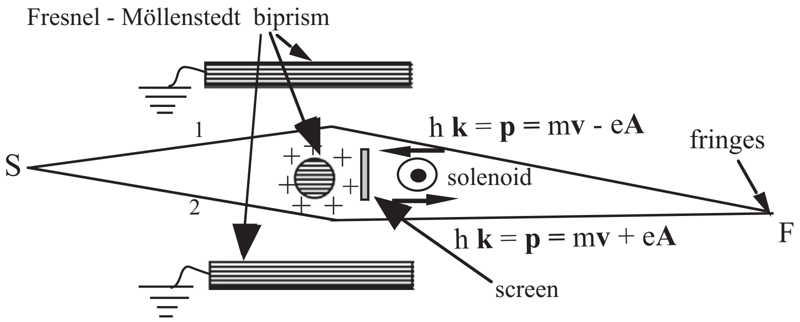

Figure 6.1 Aharonov-Bohm experiment.

We know that by virtue of Eqs. (6.22) and (6.27), or Eq. (6.28), the corresponding electromagnetic fields equal zero. The question is: how can the spin 0 photon be detected? More precisely, since these fieldless potentials are unable to generate a force, what could be observed? The answer is, of course, the phase, first characteristic of a wave. The Aharonov-Bohm effect was imagined at first by David Bohm[16] to answer this question, and to prove that contrary to a common idea, the electromagnetic potentials are not only mathematical intermediates (even if they can play this role): they are observable physical quantities. The effect had been predicted ten years earlier by Ehrenberg and Siday (1949).

6.1.7 The Effect

The idea suggested by Bohm (Aharonov and Bohm, 1959; Tonomura, 1998; Peshkin and Tonomura, 1989; Olariu and Popescu, 1985; Lochak, 1983) was to modify electron interference by a fieldless magnetic potential created by a magnetic string or by a thin solenoid orthogonal to the plan of interfering electron trajectories, as shown in Figure 6.1. The Young slits are obtained by means of a Fresnel—Möllenstedt biprism.

The solenoid must be infinitely long (in principle), so the magnetic field emanating from the extremities cannot disrupt the experiment: it is assumed in the calculations, but actually a few millimeters are sufficient because the transverse dimensions of the device are of the order of microns. This arrangement of the solenoid has led to the idea that the magnetic flux through the trajectories' quadrilateral plays an essential role. Many disagree with that idea (Lochak, 1983).

The problem of eliminating this hypothesis was elegantly solved by Tonomura (see: Peshkin and Tonomura, 1989) by substituting the rectilinear string by a microscopic torus (10 μm): one of the electron beams passes through the torus and the other outside, the magnetic lines being trapped in the torus.

[16] I know that because I was acquainted with Bohm who lived in Paris in that time.

Let us give an intuitive interpretation of the experiment. The principle is that the wave vector of an electron in a magnetic potential is given by the de Broglie wave (de Broglie, 1934c), which is a direct consequence of the identification of the principles of Fermat and of least action ($\mathbf{p}$ is the Lagrange momentum):

$$\frac{h}{\lambda}\mathbf{n} = h\mathbf{k} = \mathbf{p} = m\mathbf{v} + e\mathbf{A}. \tag{6.29}$$

It is obvious from Eq. (6.29) that interference and diffraction phenomena are influenced by the presence of a magnetic potential independent of the presence of the field because the interferences depend only on the phase. It is well known in optics: an interference figure is shifted in a Michelson interferometer by introducing a plate of glass in one of the virtual beams, which causes a phase shift and thus a change of the optical path without any additive force.

These phenomena are manifestly gauge dependent: if we add something to $\mathbf{A}$, whether a gradient or not, in the de Broglie wave λ [Eq. (6.29)], the last is modified. This is evident even in the classical de Broglie formula: $\lambda = h/mv$ when $\mathbf{A} = 0$, which is gauge dependent too, a fact often emphasized by de Broglie himself, who said, "If gauge invariance were general in quantum mechanics, the electron interferences could not exist."

In the case of the Aharonov-Bohm experiment, there is an additive phase with both interfering waves in opposite directions, which doubles the shift of the interference fringes. Let us recall a proof of the effect, independent from the fact that a potential generates forces or not (Lochak, 1983).

6.1.8 The Magnetic Potential of an Infinitely Thin and Infinitely Long Solenoid

We consider the case corresponding to the Aharonov-Bohm experiment: electrons diffracted on Young slits and falling on a magnetic solenoid orthogonal to the plane electron trajectories, according to the Figure 6.1 and, further, to the schematic shown in Figure 6.2, the solenoid is along Oz. To simplify the

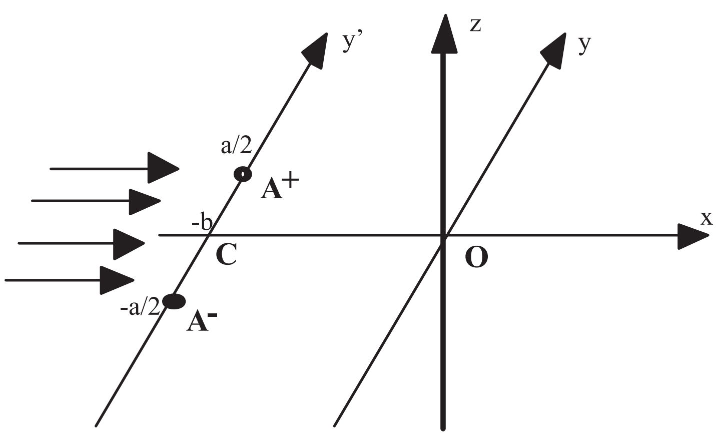

Figure 6.2 Aharonov-Bohm scheme.

calculations, we shall disregard the photon mass, which is only important in the symmetry laws, which are taken into account in all the formulas; therefore, to omit the photon mass only means to omit negligible corrections.

The electric charge of the diffracted electrons implies that they "see" the electromagnetism through the Lorentz potentials $(V, \mathbf{A})$, and thus through the equations (M): Eq. (6.21). These equations derive from the pseudo-invariant I_2. Now, there is an obvious invariant in the Aharonov-Bohm effect: the rotation angle $\varphi = \arctan y/x$ around the axis Oz. So we shall write

$$I_2 = \varepsilon k_0 \arctan (y/x), \tag{6.30}$$

where k_0 is the quantum wave number of the photon and ε a convenient dimensional constant, the value of which is not important for our calculation. Neverthless, something seems wrong here, because (y/x) is P-invariant so that, with the definition [Eq. (6.30)], I_2 seems to be a P-invariant and not a pseudoinvariant, as it needs to be in Eq. (6.22).

But this is not so because (y/x) is P-invariant only in the space $\mathbb{R}^2$: (x, y), not in the space $\mathbb{R}^3$ (x, y, z). In our case, the inversion is the P-transformation $(x, y, z) \rightarrow (-x, -y, -z)$, which implies the inversion of Oz and thus of the angle φ. So that (y/x) is really a pseudoinvariant in $\mathbb{R}^3$.

Thus, we have, by virtue of Eq. (6.22):

$$grad I_2 = k_0 \mathbf{B} \tag{6.31}$$

$$\mathbf{B}_x = -\varepsilon \frac{y}{x^2 + y^2}; \quad \mathbf{B}_y = \varepsilon \frac{x}{x^2 + y^2}; \quad \mathbf{B}_z = 0. \tag{6.32}$$

6.1.9 The Theory of the Effect

The commonly admitted theories are unnecessarily complicated (Olariu and Popescu, 1985). For the physical bases of the effect, the best is to start from the brillant book of Tonomura (1998). To find the formula of fringes, it is sufficient to take the geometrical optics approximation with the phase $\varphi = S/\hbar$ of de Broglie's wave and the principal Hamilton function S obeying the Hamilton-Jacobi equation with the potential [Eq. (6.32)]:

$$2m \frac{\partial S}{\partial t} = \left(\frac{\partial S}{\partial x} + \varepsilon \frac{y}{x^2 + y^2} \right)^2 + \left(\frac{\partial S}{\partial y} - \varepsilon \frac{x}{x^2 + y^2} \right)^2. \tag{6.33}$$

The electronic wave propagates from $x = -\infty$ to $x = +\infty$ and the Young slits A^+ and A^- (Figure 6.2) are on a parallel to Oy, at a distance $\pm a/2$ from the point C located at $x = -b$.

The pseudopotential $\mathbf{B}$ appearing in (6.30) and (6.31) is the gradient of I_2, so that $\mathbf{B}$ and I_2 satisfy up to μ_0 the equations (NM), Eq. (6.27). They are independent of t because $W = 0$.

Eq. (6.33) is immediately integrated, defining the phase as follows:

$$\Sigma = S - \varepsilon \arctan y/x \,, \tag{6.34}$$

which gives

$$2m\frac{\partial\Sigma}{\partial t} = \left(\frac{\partial\Sigma}{\partial x}\right)^2 + \left(\frac{\partial\Sigma}{\partial y}\right)^2. \tag{6.35}$$

Chosing a complete integral of Eq. (6.35) and thus of Eq. (6.32), owing to Eq. (6.33), we have

$$\Sigma = Et - \sqrt{2mE}\,(x\cos\theta_o + y\sin\theta_o) \tag{6.36}$$

$$S = Et - \sqrt{2mE}\,(x\cos\theta_o + y\sin\theta_o) + \varepsilon\arctan\frac{y}{x}, \tag{6.37}$$

or, in polar coordinates $x = r\cos\theta,\quad y = r\sin\theta$:

$$S = Et - \sqrt{2mE}\,r\cos(\theta - \theta_o) + \varepsilon\,\theta. \tag{6.38}$$

The Jacobi theorem gives the trajectories (the wave rays):

$$\begin{aligned}\frac{\partial S}{\partial\theta_o} &= \sqrt{2mE}\,(x\sin\theta_o - y\cos\theta_o) = \mu;\\ \frac{\partial S}{\partial E} &= t - \sqrt{\frac{m}{2E}}\,(x\cos\theta_o + y\sin\theta_o) = t_o.\end{aligned} \tag{6.39}$$

Finally, with[17] $E = \frac{1}{2}mv^2$ we have the motion

$$x\cos\theta_o + y\sin\theta_o = v(t - t_o). \tag{6.40}$$

We see that the *rays* (electron trajectories), defined in Eq. (6.39) are orthogonal to the moving planes but they are not orthogonal to the equal phase surfaces [Eqs. (6.37)−(6.38)] except far from the magnetic string $(x \to \infty)$, when the potential term of the order of ε becomes negligible.

Therefore, despite the presence of a potential, the electronic trajectories remain rectilinear and are not deviated, because the magnetic field equals zero by virtue of Eq. (6.22). The velocity $v = Const$ remains the one of the incident electrons because of the conservation of energy.

[17] We are obviously far from relativity.

But the diffraction of waves through the slits A^+ and A^- creates, for the electron trajectories, an interval of possible angles θ_o equal to the angles of the interference fringes, modified by the magnetic potential:

There is no deviation of the electrons, only a deviation of the angles of phase synchronization between the waves issued from A^+ and A^-. This is the Aharonov-Bohm effect, which is in accordance with the definition of the spin 0 photon [Eq. (6.22)].

It would be useless to reproduce the end of the theory of Aharonov-Bohm effect (see, for instance, Lochak, 1932b). Let us only recall the total phase-shift:

$$\Delta\varphi = \frac{\Delta S}{h} = \frac{a\theta_o}{\lambda} + \frac{2\varepsilon\xi}{h}. \tag{6.41}$$

The first term gives the standard Young fringes (the notations are those of Figure 6.2), while the second term is the Aharonov-Bohm effect: $\xi = \arctan a/2b$, which is equal to half the angle under which the Young slits are seen from the solenoid, which entails a dependence of the effect on the position of the string. One can assert that the effect decreases when the distance b increases.

We see that the theory of the Aharonov-Bohm effect is a simple consequence of the definition of the invariant in the system [Eq. (6.27)], as the invariant rotation angle around the axis of he solenoid.

6.1.10 Conclusions on the Theory of Light

We suggest a new theory of light based on four photons, as follows:

1. At first, the Einstein photon known in optics from 1905, and later identified by de Broglie (1922) as a vectorial spin 1 particle, which we call here the *electric photon,* because it interacts with the electric charges, principally with electrons.
2. A pseudovectorial spin 1 magnetic photon, analogous to the electric Einstein photon: it appears in the theory of leptonic magnetic monopoles (see: Chapters 2 and 3). The magnetic photon plays in the physics of monopoles a role exactly similar to the role played by the electric photon in the theory of electrons.
3. Two spin 0 photons (one electric and the other magnetic), related to 2 classes of respectively electric and magnetic fieldless phenomena; an example is the Aharonov-Bohm effect.
4. It must be added that in the four-photon theory of light, there are two Maxwell displacements: an electric displacement and a magnetic

displacement. Let us recall what is the Maxwell displacement[18]: at the beginning, he tried to unify the electromagnetism on the basis of several fundamental laws: the laws of Coulomb: $\nabla.\mathbf{E} = 4\pi\rho$, Ampere: $\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}$ and Faraday: $\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = 0$. But he found an incoherence between them because the third law depends on time and the other ones do not. The fact was well known and was objected to by Michael Faraday; but the critics of Maxwell went contrary to the unanimity of physicists: he considered the law of Faraday as the right one, and he decided to introduce a time dependence into the other two laws. He replaced the Coulomb law by a continuity law $\nabla.\ \mathbf{J} + \frac{\partial \rho}{\partial \mathbf{t}} = 0$, owing to which the Ampere law became $\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}$. So, he found the celebrated Maxwell equations in which appeared wave like solutions from which Maxwell found the electromagnetic theory of light and which later gave rise to the radio waves.

If we compare the (M) equations [Eq. (6.21)] of the electric photon with our (M) equations [Eq. (6.26)] of the magnetic photon, the analogy is evident: the terms of a Maxwell displacement are present in the magnetic photon, and it may be supposed that they lead to analogous physical consequences involving magnetic monopoles instead of electrons: the Aharonov-Bohm effect is a first example.

Now there is another fact that has been true for 70 years, without being pointed out until now, as far as I know. It is the fact that the de Broglie theory, based on the principle of fusion, implies automatically the displacement previously introduced by Maxwell through an external argument. The fact is hidden because the Maxwell equations make now a unit often abridged in different algebraic forms, while the displacements are more or less forgotten or rejected in the subtelties of history of science. Despite the fact that the postulate of fusion has an algebraic character, it has the advantage of unicity and of being a direct bridge between the problem of electromagnetism and the Dirac equation of the electron, the stronger equation of quantum mechanics.

It must be added that the de Broglie theory of the photon, being considered as a composite particle, gave rise to an extension to a general theory of spin particles, including gravitation (as discussed later in this chapter). We have already considered several generalizations of de Broglie's theory of

[18] See the excellent Chapter 6 of Jackson (1975). Our formalism is different from Jackson's because here, we are in the domain of quantum laws which are written in a vacuum.

light, as the magnetic photon linked to the magnetic monopole, and the Aharonov-Bohm effect, which gives rise to a new domain of electrodynamical phenomena.

The suggested theory of light is a generalization of Broglie's theory of light, with electric and magnetic photons. A new hypothesis of the present theory is that the spin 0 is condidered as a state of the photon with the same rights as the spin 1: there are not only two kinds of spin-1 photons, but also of two spin-0 photons. In other words, the photon world is divided into the same two categories as other composite quantum objects. There are orthophotons of spin 1 and paraphotons of spin 0, just as there are orthohydrogen and parahydrogen. But concerning the photon, it is a new idea, contrary to the case of orthohydrogen and parahydrogen, known for almost a century. This is why many questions still remain asked, such as the following:

- What happens with the spin-0 photons in the thermodynamical equilibrium?
- We have seen that paraphotons, being fieldless, are unable to create a force; so, are they able to produce something like a photoelectric effect? It seems not.
- More generally, are there true quantum wave-particle objects, or "pure-phases," pure potentials without particles? (*Pace* Louis de Broglie!)
- There are arguments in favor of some of these hypotheseses. For instance, the existence of a magnetic spin 1 photon is confirmed by the experiments on the leptonic monopole. Until now, the Aharonov-Bohm effect was a remarkable but isolated orphan effect. Here, it is integrated in a general theory. This is fine, but a question remains: is this effect exceptional, or is it a sample of a "class" of new phenomena? The equations define such a class mathematically, but it must be experimentally proved that such phenomena really exist as several physical effects. We have predicted at least one such effect: the Aharonov-Bohm effect with magnetic monopoles, but it is not yet observed and not yet calculated with all the details.

6.2 HAMILTONIAN, LAGRANGIAN, CURRENT, ENERGY, SPIN

6.2.1 The Lagrangian

Now, let us go back to the 16-line column wave function and the canonical form Eq. (6.12), keeping only (C) because (D) is deduced from it:

$$\frac{1}{c}\frac{a_4+b_4}{2}\frac{\partial\phi}{\partial t}=\frac{b_4a_k+a_4b_k}{2}\frac{\partial\phi}{\partial x_k}+i\frac{\mu_0 c}{\hbar}a_4b_4\phi. \tag{6.42}$$

Note the presence of $(a_4+b_4)/2$ in the factor of $\partial/\partial t$, so it seems unescapable that coherent definitions for tensor densities would be obtained. The Hamiltonian operator is

$$H=i\hbar\left[\frac{b_4a_k+a_4b_k}{2}\frac{\partial}{\partial x_k}+i\frac{\mu_0 c}{\hbar}a_4b_4\right] \tag{6.43}$$

and the Lagrangian density is (with $\phi^+=\phi\ \ (h.c.)$)

$$L=-i\hbar c\left[\phi^+\left(\frac{1}{c}\frac{a_4+b_4}{2}\frac{\partial\phi}{\partial t}-\frac{b_4a_k+a_4b_k}{2}\frac{\partial\phi}{\partial x_k}-i\frac{\mu_0 c}{\hbar}a_4b_4\phi\right)+c.c.\right]. \tag{6.44}$$

6.2.2 The Current Density Vector

The general formula

$$J_\mu=\frac{i}{\hbar}\left[\frac{\partial L}{\partial\phi_{,\mu}}\phi-\frac{\partial L}{\partial{\phi^+}_{,\mu}}\phi^+\right] \tag{6.45}$$

gives, with Eq. (6.44),

$$J_k=-c\phi^+\frac{b_4a_k+a_4b_k}{2}\phi;\ \ J_4=ic\rho;\ \ \rho=\phi^+\frac{a_4+b_{4k}}{2}\phi. \tag{6.46}$$

Therefore, *$\int\rho dv$ is not definite-positive*. But on the other hand, we shall find a *definite-energy* $\int\rho W dv\geq 0$, contrary to what happens in the Dirac electron. This result will be generalized in the general theory of particles with *spin* $=\frac{n}{2}$.

In terms of electromagnetic quantities, Eq. (6.45) is given by the Geheniau formulas, with two kinds of terms corresponding to spin 1 and spin 0 in the case of an electric photon (de Broglie, 1943). Here, until the end of the next section, we give only the translation of the formulas in the electric case (they were not translated until now in the magnetic case):

$$\begin{aligned}\mathbf{J}&=\frac{i}{\hbar c}[\mathbf{A}^*\times\mathbf{H}+\mathbf{H}^*\times\mathbf{A}+V^*\mathbf{E}-\mathbf{E}^*V]+\frac{c}{4}\left(I_2^*\mathbf{B}+\mathbf{B}^*I_2\right)\\ \rho&=\frac{i}{\hbar c}[(\mathbf{A}^*.\,\mathbf{E})-(\mathbf{E}^*.\,\mathbf{A})]+\frac{1}{4}\left(I_2^*W+W^*I_2\right).\end{aligned} \tag{6.47}$$

For the energy tensor, we have the general formula

$$T_{\mu\nu} = -\frac{\partial L}{\partial \phi_{,\mu}}\phi_{,\mu} - \frac{\partial L}{\partial \phi^{+}_{,\mu}}\phi^{+}{}_{,\mu} + L\delta_{\mu\nu}\ , \tag{6.48}$$

with the Lagrangian [Eq. (6.44)], which gives

$$\begin{aligned}
T_{ik} &= -\frac{i\hbar c}{2}\left[\phi^{+}\frac{b_4 a_k + a_4 b_k}{2}\frac{\partial \phi}{\partial x_k} + \frac{\partial \phi^{+}}{\partial x_k}\frac{b_4 a_k + a_4 b_k}{2}\phi\right] \\
T_{i4} &= \frac{\hbar}{2}\left[\phi^{+}\frac{b_4 a_k + a_4 b_k}{2}\frac{\partial \phi}{\partial t} + h.c.\right];\quad T_{4i} = -\frac{\hbar c}{2}\left[\phi^{+}\frac{a_4 + b_4}{2}\frac{\partial \phi}{\partial x_i} + h.c.\right] \\
T_{44} &= -w = i\hbar\frac{i\hbar}{2}\phi^{+}\frac{a_4 + b_4}{2}\frac{\partial \phi}{\partial t} = -\phi^{+}H\phi.
\end{aligned} \tag{6.49}$$

In the electromagnetic form, we have

$$\begin{aligned}
T_{\mu\nu} &= \frac{1}{2}\left(F_{\mu\lambda}\frac{\partial \mathbf{A}_\lambda}{\partial x_\nu} - \mathbf{A}_\lambda\frac{\partial F_{\lambda\mu}}{\partial x_\nu}\right) - \frac{i\hbar}{8}\left(I_2^{*}\frac{\partial \mathbf{B}_\lambda}{\partial x_\nu} + \mathbf{B}^{*}_{\mu}\frac{\partial I_2}{\partial x_\nu}\right) + \text{c.c.} \\
&\text{where :}\quad F_{\mu\lambda} = \frac{\partial \mathbf{A}_\mu}{\partial x_\nu} - \frac{\partial \mathbf{A}_\nu}{\partial x_\mu}
\end{aligned} \tag{6.50}$$

In particular, the energy density ρW takes the form

$$\begin{aligned}
T_{44} = \frac{1}{2c}&\left[\left(\mathbf{A}^{*}.\frac{\partial \mathbf{E}}{\partial t}\right)\left(\mathbf{E}^{*}.\frac{\partial \mathbf{A}}{\partial t}\right) + \left(\mathbf{A}.\frac{\partial \mathbf{E}^{*}}{\partial t}\right)\left(\mathbf{E}.\frac{\partial \mathbf{A}^{*}}{\partial t}\right)\right] \\
&+ \frac{i\hbar c}{2}\left[\left(I_2^{*}\frac{\partial W}{\partial t}\right)\left(W^{*}\frac{\partial I_2}{\partial t}\right) - \left(I_2\frac{\partial W^{*}}{\partial t}\right)\left(W\frac{\partial I_2^{*}}{\partial t}\right)\right].
\end{aligned} \tag{6.51}$$

The tensor $T_{\mu\nu}$ is often symmetrized, putting $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$, but there are strong arguments in favor of the nonsymmetric tensor (Costa de Beauregard, 1943; de Broglie, 1943).

In addition, we can find other tensors, the integrals of which are equal to the integral of the precedings (they differ by a divergence). One of these tensors is[19]

[19] The factor μ_0 is surprising but according to Eq. (6.20), it disappears from the fields and potentials.

$$M_{ik} = M_{ki} = \mu_0 c^2 \phi^+ \frac{a_i b_k + a_k b_i}{2} \phi; \quad M_{i4} = M_{4i} = -\mu_0 c^2 \phi^+ \frac{a_i + b_i}{2} \phi;$$
$$M_{44} = \mu_0 c^2 \phi^+ \phi; \quad (i, k = 1, 2, 3). \tag{6.52}$$

This is a Maxwell-type tensor because we find, in electromagnetic terms, for the electric photon:

$$\begin{aligned} M_{i4} &= (\mathbf{E}.\, \mathbf{H}^*)_i + (\mathbf{E}^*.\, \mathbf{H})_i - k_0^2 \left(V^* \mathbf{A}_i + V \mathbf{A}_i^* \right); \\ M_{44} &= |\mathbf{E}|^2 + |\mathbf{H}|^2 - k_0^2 \left(|\mathbf{A}|^2 + |V|^2 \right). \end{aligned} \tag{6.53}$$

We recognize the Maxwellian form, up to the mass terms, and we find

$$\int M_{\mu\nu}\, d\tau = \int T_{\mu\nu}\, d\tau. \tag{6.54}$$

6.2.3 The Photon Spin

Let us express the angular momentum with the nonsymmetric tensor $T_{\mu\nu}$:

$$m_{ik} = -\frac{i}{c} \int [x_i T_{4k} - x_k T_{4i}]\, d\tau \quad (i, k = 1, 2, 3), \tag{6.55}$$

where m_{ik} is *not* a constant of motion. But, as in Dirac's theory, we find a constant of motion $\mathrm{m}'_{\mathrm{ik}}$ if we add a convenient term of spin:

$$m'_{ik} = m_{ik} + S_{ik} \tag{6.56}$$

$$S_{ik} = i\hbar \int \phi^+ \frac{b_4 a_i a_k + a_4 a_i b_k}{2} \phi \quad (i, k = 1, 2, 3). \tag{6.57}$$

The dual $s_j = \varepsilon_{jik} S_{ik}$ of this tensor in $\mathbb{R}^3$ is a pseudovector. Analogous with the Dirac spin, we find a *space-time pseudovector,* by adding a time component:

$$s_4 = c\hbar \int \phi^+ \frac{b_4 a_1 a_2 a_3 + a_4 b_1 b_2 b_3}{2} \phi. \tag{6.58}$$

Now if we introduce into Eq. (6.55) the tensor $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$ instead of the tensor $T_{\mu\nu}$, we find the new momentum, which is equivalent to Eq. (6.57):

$$m'_{ik} = -\frac{i}{c} \int \left[x_i T_{(4k)} - x_k T_{(4i)} \right] d\tau \quad (i, k = 1, 2, 3). \tag{6.59}$$

Of course, this is a conservative tensor. The difference between this and the theory of the electron is that the eigenvalues of the matrices in the integrals [Eq. (6.57)] are -1, 0, and 1, instead of $\pm\frac{1}{2}$. We have a particle of maximum spin 1. The space-time pseudovector $s_\mu = \{\mathbf{s}, s_4\}$ has the following form in terms of electromagnetic quantities in the case of the electric photon:

$$\mathbf{s} = \frac{1}{c}[\mathbf{E}^* \times \mathbf{A} - \mathbf{A}^* \times \mathbf{E} + V^*\mathbf{H} + \mathbf{H}^* V]; \quad s_4 = \frac{1}{c}[\mathbf{A}^*.\,\mathbf{H} + \mathbf{H}^*.\,\mathbf{A}]. \tag{6.60}$$

Only terms corresponding to spin 1 appear in that formula. The terms corresponding to spin 0 vanish because $I_1 = 0$; this fact is not astonishing because $\mu_0 \neq 0$ [see Eq. (6.22)]. If, we had started from Eq. (6.11) instead of Eq. (6.12), we should be nearer to Dirac's theory. Now consider the orbital momentum operator:

$$\mathbf{M}_{op} = \mathbf{r} \times \mathbf{p}. \tag{6.61}$$

This operator is not an integral of the motion, but we can find a commutating operator by adding to M_{op} the new spin operators:

$$\mathbf{S} = \left\{ -i\hbar\left(\frac{a_2a_3 + b_2b_3}{2}\right),\ -i\hbar\left(\frac{a_3a_1 + b_3b_1}{2}\right),\ -i\hbar\left(\frac{a_1a_2 + b_1b_2}{2}\right)\right\}, \tag{6.62}$$

which must be completed by

$$S_4 = -\frac{i\hbar}{2}(a_1a_2a_3 + b_1b_2b_3), \tag{6.63}$$

which gives with $\mathbf{S}$ a relativistic quadrivector. The space components of $\mathbf{S}$ satisfy the spin commutation relations, and finally these definitions will be used in the generalized theory of fusion.

6.2.7 Relativistic Noninvariance of the Decomposition Spin 1−Spin 0

The spin operators $s_j = \varepsilon_{jik}S_{ik}$ satisfy the commutation rules of an angular momentum and they have the eigenvalues $\{-1,\ 0,\ 1\}$. The total spin $\mathbf{s}^2$ has the eigenvalues $l(l+1) = (2,0)$, corresponding to $l = 1, 0$.

In the case of a plane wave in Eqs. (6.21) and (6.22) and Eqs. (6.29) and (6.24), one can show that the group of equations (M) is associated with $l = 1$, with projections $s = -1, 0, +1$ on the direction of propagation of the wave: $s = -1 \Leftrightarrow$ *right circular wave*, $s = +1 \Leftrightarrow$ *left circular wave*. For

$s = 0$, we have in both cases a small longitudinal *electric* wave (due to the mass) for the electric photon, and a small longitudinal *magnetic* wave for the magnetic photon. The group (NM) is associated with $l = 0$.

So we can speak of (M) as a "spin 1 particle" and of (NM) as a "spin 0 particle." However, de Broglie made an important distinction (de Broglie, 1943, Chapter 8): *although the equations (M) and (NM) are relativistically invariant, the separation between them is not covariant because it is based on the eigenvalues of the total spin-operator* $\mathbf{s}^2 = s_1^2 + s_2^2 + s_3^2$, which is not a relativistic invariant. The correspondence between the field values and the eigenvalues of $\mathbf{s}^2$ is as follows:

1. For the electric photon:

$$\begin{array}{cccc c cccccc} \mathbf{A} & V & \mathbf{E} & \mathbf{H} & & I_1 & \mathbf{B} & W & I_2 & \mathbf{E}' & \mathbf{H}' \\ 2 & 0 & 2 & 2 & & 0 & 2 & 0 & 0 & 2 & 2; \end{array} \tag{6.64}$$

2. For the magnetic photon:

$$\begin{array}{cccc c cccccc} \mathbf{B}' & W' & \mathbf{H}' & \mathbf{E}' & & I_2 & \mathbf{A}' & V' & I_1 & \mathbf{H} & \mathbf{E} \\ 2 & 0 & 2 & 2 & & 0 & 2 & 0 & 0 & 2 & 2, \end{array} \tag{6.65}$$

In both cases, the first group corresponds to the (M) equations and the second group to (NM). We can note, when passing from Eq. (6.64) to Eq. (6.65), the following exchanges:

- Between potentials $\mathbf{A}$, V and pseudopotentials $\mathbf{B}', W'$
- Between fields $\mathbf{E}$, $\mathbf{H}$ and anti-fields $\mathbf{E}'$, $\mathbf{H}'$ [we know that $\mathbf{E}', \mathbf{H}' = 0$ in Eq. (6.64) and $\mathbf{E}, \mathbf{H} = 0$ in Eq. (6.65)]
- Between I_1 and I_2, in the group (NM) ($I_1 = 0$ in Eq. (6.64) and $I_2 = 0$ in Eq. (6.65)

The most important fact is that there are in both groups (M) and (NM), field quantities with $\mathbf{s}^2 = 2$ and $\mathbf{s}^2 = 0$, and thus spin 1 and spin 0 components: there is no true separation between the values of spin. De Broglie has shown (for both photons) that the separation only occurs in the proper system, as follows:

1. Because for the electric photon, the potential $(\mathbf{A}, V)$ is *spacelike*, and the pseudopotential $(\mathbf{B}, W)$ is *timelike*, so that V and $\mathbf{B}$ disappear from (6.51), and only $\mathbf{s}^2 = 2$ remains in (M), conversely, only $\mathbf{s}^2 = 0$ remains in (NM) because we know that $\mathbf{E}' = \mathbf{H}' = 0$.
2. For the magnetic photon, the same thing happens because this case follows from the preceding by multiplying an electric solution by γ_5, exchanging polar and axial quantities:

$$(\mathbf{E}, \mathbf{H}) \leftrightarrow (\mathbf{H}', \mathbf{E}'); \quad (V, \mathbf{A}) \leftrightarrow (W, \mathbf{B}); \quad (I_1, I_2) \leftrightarrow (I_2, I_1). \tag{6.66}$$

Therefore, the potential $(\mathbf{A}, V)$ becomes *timelike* and the pseudopotential $(\mathbf{B}, W)$ becomes *spacelike*. And we have once more in the proper frame $\mathbf{s}^2 = 2$ in (M) and $\mathbf{s}^2 = 0$ in (NM), taking into account that we have $\mathbf{E} = \mathbf{H} = \mathbf{0}$ instead of $\mathbf{E}' = \mathbf{H}' = \mathbf{0}$.

In conclusion, the (M) and (NM) groups of equations cannot be rigorously separated, except in the proper frame, and they must be considered as forming one block, for two reasons:

1. The difficulty of separating spin 1 and spin 0 means that the composite photon cannot be considered as a spin 1 particle, but as a particle with a *maximum spin 1*, just as a two-electron atom or a two-atom molecule. It is noteworthy that the proper state in which the 1-components and 0-components are separated is obviously the same for both components.
2. On the contrary, the presence of two photons (electric and magnetic) is inscribed in the very structure of the theory; their separation is covariant and more radical than the separation of spin-states. The simultaneous presence in (M) and (NM) equations of potentials and pseudopotentials and of fields and anti-fields (even if half of them equal zero), and the "migration" of these quantities from one group of equations to the other according to the type of photon constitute another link.

Of course, at the present stage of the problem, a question remains: what is this spin 0 component, physically? It could seem that all these questions are raised by the hypothesis $\mu_0 \neq 0$. Of course, they could be avoided if we admit that $\mu_0 = 0$. But it would be certainly a bad idea to shield the theory from a physical difficulty by a formal condition, at the expense of a more synthetic structure, as was shown previously. A better answer will be given later in this chapter, by the simple fact that the spin 0 component is a photon state that plays a physical role, just as the spin 1 state, and they must be included in the same global theory of light.

6.2.8 The Problem of a Massive Photon

We have seen that many features of de Broglie's theory of the photon, including its logical coherence, are due to the hypothesis $\mu_0 \neq 0$. But even if μ_0 is small, it implies many differences with ordinary electromagnetism. These differences were examined in a number of papers (e.g., de Broglie, 1936, 1940–1942, 1943; Costa de Beauregard, 1997b, 1983; Borne, Lochak, & Stumpf, 2001; Lochak, 1995b, 2000, 2002, 2007b; Lochak and Costa de Beauregard, 2000).

6.2.9 Gauge Invariance

Obviously, the common phase invariance disappears if $\mu_0 \neq 0$, which then calls for some comments:

- First, why do we find in de Broglie's theory of light the *Lorentz gauge* as a field equation? Simply because it is the only relativistically invariant, linear differential law of the first order: it was the only possibility.
- There remain some practical problems. The relations between potentials and fields show that they are of the same order of magnitude. The mass terms are thus of k_0 order: that is, very small. Therefore, in general, the gauge symmetry remains, up to a negligible error, and we can still choose with good approximation the convenient gauge for most practical problems, provided that physics does not impose a particular choice.
- In the present theory, the potentials are deducible from the fields, thus from observable phenomena: they are no longer mathematical fictions, but physical quantities. It must be noted that such a conception was already devised by Maxwell himself (Maxwell, 1873).

This is important for zero-field phenomena only because of a potential, as is the case for the Aharonov-Bohm effect. The fact that this last effect is not gauge invariant is not an objection because we know other physical quantities that are only partially defined by some effects but exactly defined by others: for instance, energy is defined by spectral laws up to an additive constant, but exactly fixed by relativistic effects.

De Broglie gave another example of a physically defined potential: the electron gun (de Broglie, 1943), in which the potential V between the electrodes is exactly defined for several reasons:

1. The *measurable* velocity of the emerging electron is given by the increase of energy, which is equal to eV.
2. The phase of the wave associated with the electron is relativistically invariant only if the frequency and the phase velocity obey the classical de Broglie formula, which imposes the gauge of V (as already noted).
3. The fundamental reason is that the intertia of energy does not allow an arbitrary choice of the origin of electrostatic potentials, which actually are not gauge invariant. They are physical quantities, related to mesurable effects. More recently, Costa de Beauregard and Lochak published many other impressive experimental examples, in favor of the physical sense of electromagnetic potentials.

After several attempts, de Broglie and other authors supposed that the Dirac particles that were constituted by fusion photons and gravitons were *neutrinos*. For a long time, the neutrino was considered a massless particle, with arguments based on gauge invariance, separation of chiral components, etc. But new theoretical arguments based on hypothetical oscillations between different kinds of neutrinos, the subsequent need of coupling constants, and some experimental evidence pointed to a possible neutrino mass. If this is confirmed by facts, de Broglie's fusion theory will have as a consequence the prediction of a photon and a graviton mass, which will become in turn a credible idea. It must be confessed that the leptonic monopole theory (which is due to the author of these lines, who is a member of the same theoretical school) is not in agreement with the last opinion. Nevertheless, it must be remembered (see Chapter 4 of this part of the book) that there is also a theory of massive magnetic monopoles with the same symmetries, but it is a nonlinear theory, different from the present one.

6.2.10 Vacuum Dispersion

If $\mu_0 \neq 0$, we can write

$$h\nu = \frac{\mu_0}{\sqrt{1-\frac{v^2}{c^2}}} \rightarrow v = c\sqrt{1-\frac{\mu_0^2 c^2}{h^2\nu^2}} \quad (v = \text{group velocity}). \tag{6.67}$$

Thus, the vacuum must be dispersive, which was not yet observed, but it may be stressed that the supposed value $\mu_0 < 10^{-45}$g implies a Compton wavelength: $\lambda_c > 10^8$cm $= 10^3$km, so that the substitution of the Coulomb potential $\frac{1}{r}$ by the corresponding Yukawa potential $\frac{e^{-k_0 r}}{r}$ has a very small practical incidence, as with other numerical quantities. But the consequences of the symmetry laws are important.

Another question is that one could in principle observe a photon with a velocity smaller than c in the vacuum. In de Broglie's time, his estimations proved that it was impossible if $\mu_0 < 10^{-45}$g (de Broglie, 1936, 1940–1942). Nevertheless, with the progress of experimental physics, such a possibility must be reexamined.

6.2.11 Relativity

Practically, the velocity predicted for the photon is so near from c, that the difference has not any consequence (at least at the present level of knowledge). But the problem is: how shall we built the theory of relativity? De Broglie's answer was one of his favorite jokes: "Light is not obliged to go with the velocity of light." In other words: we need, in relativity, a maximum invariant velocity, but we do not need this velocity to be the velocity of light. It only happens that, in a vacuum, the velocity of light is close to it.

6.2.12 Blackbody Radiation

In a given unit-volume, there are $dn_\nu = \frac{4\pi\nu^2}{c^3} d\nu$ stationary waves of light in an elementary interval of frequencies, and we must have twice this number because of the transversality of light waves, which gives a factor of 8 in Planck's law of blackbody radiation. But if $\mu_0 \neq 0$, it seems that we must multiply by 3 (instead of 2) because there is a longitudinal electric-component that gives 12 in Planck's law.

But this is wrong. The answer is actually that if we apply the formula for energy, it is shown that the longitudinal part of the field (so as the one, corresponding to potentials) is of the order of k_0—that is, it is negligible (de Broglie, 1936, 1940—1942), so that it takes no part in the observed equilibrium and the factor 8 is the right one. This argument, given by de Broglie, was later independently confirmed by Bass and Schrödinger (1955).

6.2.13 A Remark on Structural Stability

A physical theory has (at least) three truth-criteria: experiment, logical consistency, and structural stability. The first two points are evident, while the third is less so. It means that a theory must have a sufficiant adaptability to withstand slight experimental deviations without its mathematical frame being destroyed.

Actually, most physical theories are too rigid and have structural *unstabilities:* for instance, Hamiltonian dynamics is structuraly unstable because its formalism does not allow the slightest dissipation. This means that the condition of structural stability, despite the strength of the argument and the high authority of the signatures, cannot be respected by all theories. But, at least, one must eliminate arithmetical conditions or too precise symmetries, which could not be verified experimentally.

An example is the mass of the photon. It is proved experimentally that the mass is *small,* but it cannot be proved that this mass is *exactly zero* because it would be an *arithmetical condition*. In other words, electromagnetic gauge invariance—as a law of symmetry—may be proved approximately, not exactly.

It would be extremely worrying if electromagnetism needed exactly zero mass and gauge invariance[20]. And this is not the case, but by virtue of Broglie's theory of photons, the smallness μ_0 implies negligible deviations in the experimental facts.

6.3 THEORY OF PARTICLES WITH MAXIMUM SPIN n

6.3.1 Generalization of the Theory

The general theory is the subject of the second part of de Broglie (1943). Here, we are giving only a short summary—even shorter than for the case of spin 1. The link with the monopole will appear later.

6.3.2 Generalized Method of Fusion

Extending Eq. (6.7), the fusion of n Dirac equations gives a generalization of Eq. (6.8):

$$\frac{1}{c}\frac{\partial\phi_{ikl\ldots}}{\partial t}=a_k^{(p)}\frac{\partial\phi_{ikl\ldots}}{\partial x_k}+i\frac{\mu_0 c}{\hbar}a_4^{(p)}\phi_{ikl\ldots}\quad(p=1,2,\ldots,n).\tag{6.68}$$

Thus, we have n equations instead of 2, and a 4^n component wave function (a spinor of nth rank) instead of 16 components for the photon. And there are $4n$ matrices $(a_r^{(p)})$ with 4^{2n} elements:

$$\left(a_r^{(p)}\right)_{ik\ldots opq\ldots,i'k',\ldots o'p'q'\ldots}=\delta'_{ii}\delta'_{kk}\ldots\delta'_{oo}(\alpha_r)'_{pp}\delta'_{qq}\ldots\tag{6.69}$$

They obey the Eq. (6.10) relations:

$$a_r^{(p)}a_s^{(p)}+a_s^{(p)}a_r^{(p)}=2\delta_{rs};\,a_r^{(p)}a_s^{(q)}-a_s^{(q)}a_r^{(p)}=0\ \ (\text{if }p\neq q).\tag{6.70}$$

The same problem as in Eq. (6.8), occurs here: there are n times too many equations (whereas for the photon, we had twice as many). We have indeed $n4^n$ equations for 4^n components of the wave function. The answer is almost the same.

6.3.3 "Quasi-Maxwellian" Form

We shall proceed as in §**6.3.1).** But we first put the following expression:

[20] A theory of A. Eddington was based on 16 degrees of freedom and needed the *exact formula* $\frac{1}{\alpha}=\frac{16(16+1)}{2}+1=137$ (α = fine structure constant). Unfortunately, measurement gives $\frac{1}{\alpha}=137.036\ldots$

$$F^{(p)} = a_k^{(p)} \frac{\partial}{\partial x_k} + i \frac{\mu_0 c}{\hbar} a_4^{(p)}. \tag{6.71}$$

We have the relation

$$F^{(p)} F^{(q)} = F^{(q)} F^{(p)}, \quad \forall p, q; \quad \left(F^{(p)}\right)^2 = \Delta - k_0^2\,, \tag{6.72}$$

which implies that the wave obeys the Klein-Gordon equation. Now, Eq. (6.68) takes the form

$$\frac{1}{c} \frac{\partial \phi}{\partial t} = F^{(p)} \phi; \quad p = 1, 2, \ldots n. \tag{6.73}$$

By adding these equations, we find a new evolution equation generalizing the (A) expression in Eq. (6.11):

$$\text{(A)}\ \frac{1}{c} \frac{\partial \phi}{\partial t} = F\phi; \quad F = \frac{1}{n} \sum_{p=1}^{n} F^{(p)}. \tag{6.74}$$

Now, subtracting the expressions Eq. (6.73) from each other in a convenient way, we can eliminate the time derivatives and find $(n - 1)$ "condition equations." This may be done in many ways. For instance, we can choose the following system, similar to the (B) expression in Eq. (6.11):

$$\text{(B)}\ B^{(p)} \phi = \frac{F^{(1)} - F^{(p)}}{2} \phi = 0 \quad (p = 2, 3, \ldots, n). \tag{6.75}$$

It is easy to prove that the new systems (A) and (B) are equivalent to Eq. (6.68) or Eq. (6.72). Owing to Eq. (6.69), one can see that F and B commute, but their product doesn't equal zero, contrary to what happened with the operators on the right-hand side of Eq. (6.11) in the special case $n = 2$:

$$B^{(p)} F = F B^{(p)} \neq 0. \tag{6.76}$$

This means that, contrary to Eq. (6.11), we cannot use Eqs. (6.74) and (6.75) to prove that the conditions (B) are deducible from the evolution equation (A). However, as a consequence of Eq. (6.63), the left-hand sides $B^{(p)}\ \Phi$ of Eq. (6.75) are solutions of Eq. (6.73), so that, if the conditions (B) are satisfied at an initial time $t = 0$, they are satisfied at all time.

On the other hand, we can prove the compatibility of the $(n - 1)$ equations (B), so that the compatibility of the system (6.67)—or, equivalently of Eqs. (6.73) and (6.74)—is proved.

6.3.4 The Density of Quadri-current

Generalizing the case of maximum spin 1, de Broglie introduced another set of matrices (de Broglie, 1943):

$$B_4^{(p)} = a_4^{(1)} a_4^{(2)} \ldots a_4^{(p-1)} a_4^{(p+1)} \ldots a_4^{(n)}. \tag{6.77}$$

Each is the product of all the values of $a_4^{(i)}$ except the one corresponding to the index p. The quadri-current density is as follows, and it is easy to verify that it is conservative:

$$J_k = -c\phi * \frac{1}{n} \sum_{p=1}^{n} a_k^p B_4^p \phi; \quad \rho = \phi * \frac{1}{n} \sum_{p=1}^{n} B_4^p \phi; \quad \frac{\partial \rho}{\partial t} + \partial_k J_k = 0. \tag{6.78}$$

Generalizing a remark made in § 5.2, it is interesting to examine the ρ *density*. Following de Broglie, we shall do it in the case of the plane wave. Let us note, by the way, that it is not difficult to calculate a plane wave for a particle of maximum spin $n/2$: the phase is evident, and the amplitudes are given by the n products of 4 amplitudes of n Dirac plane waves, which gives 2^n constants restricted by the fusion conditions. The calculation is rather long (de Broglie, 1940–1942), but the result is simple. We find

$$\overline{Q} = \rho\, Q\,, \tag{6.79}$$

with

$$\rho = \left(\frac{\mu_0 c^2}{W}\right)^{n-1} |\phi|^2 \; (\mu_0 \text{ and } n = \text{mass and number of spin } 1/2 \text{ particles}) \tag{6.80}$$

From this, we see the following:

- If n is odd, the sign of ρ is definite-positive, as in the case $n = 1$ of a Dirac electron.
- If n is even, ρ has the same sign as energy, and it is indefinite: it was the case for a photon (spin 1), and it is the case for a graviton (spin 2).

It is interesting to note, with de Broglie, the curious presence, in Eq. (6.80), of the $(n - 1)$th power of the Lorentz contraction, which means that the density ρ, integrated over a volume ($\int \rho d\mathrm{v}$), will be contracted exactly n times (the number of elementary spin 1/2 particles). The exception is the Dirac particle, for which $n - 1 = 0$, so that the factor disappears and the integral is contracted only by the integration-volume itself. De Broglie conjectured that this factor is perhaps an echo of a hidden spatial structure of the composite particle, which we can describe only as a point in the present state of linear quantum mechanics.

6.3.5 The Energy Density

We begin with an elementary calculation of the *energy density* using the preceding density ρ for a plane wave. The definition of the density ρ means that all the mean values are obtained by the integration of a physical quantity multiplied by ρ.

The energy density is thus obtained (in the case of a plane wave) owing to Eq. (6.80):

$$\rho W = \left(\frac{\mu_0 c^2}{W}\right)^{n-1} W|\phi|^2. \tag{6.81}$$

Here, the power of W is not $(n-1)$ but $(n-2)$, so that we find a result opposite to the result for ρ:

- If n is odd, ρW has the same indefinite sign as energy: it was the case for $n = 1$, for the Dirac electron.
- If n is even, the sign of ρW is definite-positive, as it was for the photon and as it will be for the graviton. This is confirmed by more sophisticated calculations using the energy tensor density.

We shall introduce two classes of tensors. The first, named "corpuscular" by de Broglie, is given by the receipts of quantum mechanics. The second class, called by de Broglie "of type M" (with M standing for "Maxwell"), is wider and is inspired by electromagnetism.

6.3.6 The "Corpuscular" Tensor

We use the B matrices definited in Eq. (6.77) with the following notation:

$$\rho U_i^{(p)} = a_i^{(p)} \ \ (i = 1, 2, 3,), \ \ U_i^{(p)} = 1; \ \ (p = 1, 2, ..., n). \tag{6.82}$$

The tensor is then (de Broglie, 1943), generalizing the spin 1 case:

$$T_{\mu\nu} = T_{\nu\mu} = \frac{\hbar c}{4in} \sum_{p=1}^{n} \begin{bmatrix} \phi^* U_\mu^{(p)} B_4^{(p)} \dfrac{\partial\phi}{\partial x_\nu} - \dfrac{\partial\phi *}{\partial x_\nu} U_\mu^{(p)} B_4^{(p)} \phi \\ +\phi^* U_\nu^{(p)} B_4^{(p)} \dfrac{\partial\phi}{\partial x_\mu} - \dfrac{\partial\phi *}{\partial x_\mu} U_\nu^{(p)} B_4^{(p)} \phi \end{bmatrix}. \tag{6.83}$$

We verify its conservation by virtue of the equations

$$\partial_\nu T_{\mu\nu} = \partial_\mu T_{\mu\nu} = 0. \tag{6.84}$$

It is interesting to verify that the tensor takes the form that is to be expected for a plane wave, and we find indeed the following matrix for its components ($\mathbf{p}$ = momentum, $\mathbf{v}$ = group velocity):

$$\begin{Bmatrix} \rho p_1 v_1 & \rho p_1 v_2 & \rho p_1 v_3 & \rho p_1 c \\ \rho p_2 v_1 & \rho p_2 v_2 & \rho p_2 v_3 & \rho p_2 c \\ \rho p_3 v_1 & \rho p_3 v_2 & \rho p_3 v_3 & \rho p_3 c \\ \rho p_1 c & \rho p_2 c & \rho p_3 c & \rho W \end{Bmatrix}. \tag{6.85}$$

In particular T_{44} is the quantity given in Eq. (6.81).

6.3.7 The "type M" Tensors

At first, we shall generalize the formula [Eq. (6.77)] by the definition of a set of operators of rank m:

$$B_4^{(pq\ldots)} = a_4^1 a_4^2 \ldots a_4^{p-1} a_4^{p+1} \ldots a_4^{q-1} a_4^{q+1} \ldots . \tag{6.86}$$

This is the product of all the values of a_4^r ($r = 1, 2, \ldots, n$) except those for which r is equal to one of the m indices $p, q \ldots$ of B. Using of these operators and Eq. (6.81), we define a set of tensors of rank m (de Broglie, 1934c):

$$M_m = \mu_0 c^2 \phi^* \frac{\sum_{pq\ldots} U_i^{(p)} U_j^{(q)} \ldots B_4^{(pq\ldots)}}{a_n^m} \phi; \quad \left(a_n^m = \frac{n!}{(n-M)!} \right). \tag{6.87}$$

These tensors are obviously symmetric, but we keep only those values of the rank $m = 2r$ that are even. Thus, we have defined (for a particle of maximum spin n) $n/2$ tensors if n is even and $(n-1)/2$ tensors if n is odd. Finally, we contract each tensor of rank $2r$, over $2r - 2$ indices, which gives a number equal to the half of the greatest even number contained in n of tensors of rank 2, according to the following formula:

$$M_{ij}^{(r)} = \sum_{ijkl\ldots}^{4} M_{ijkl\ldots}^{kl\ldots}. \tag{6.88}$$

We must remember that, applying the receipt to real space-cordinates, we must change the sign when indices 1, 2, and 3 go up or down.

These tensors were defined by de Broglie as tensors "of type M." By virtue of the general equations [Eq. (6.87)], we have, just as for the tensor T:

$$\partial_\nu M_{\mu\nu}^{(r)} = \partial_\mu M_{\mu\nu}^{(r)}, \tag{6.89}$$

and we have $n/2$ tensors $M^{(r)}$ of rank 2 if n is even and $(n-2)/2$ tensors if n is odd.

A priori, each conservative tensor may be considered as an impulse-energy tensor, and it may be shown that, *for a plane wave*, $\forall\ r$ every tensor $M_{\mu\nu}^{(r)}$ gives exactly the table of components [Eq. (6.83)]. This is not true for other solutions, but it remains true of the integrals:

$$\int T_{\mu\nu} d\tau = \int M_{\mu\nu}^{r} d\tau; \quad \forall r. \tag{6.90}$$

6.3.8 Spin

Starting from Eq. (6.72)—the generalization of Eq. (6.11)—we have the same orbital operator, and the spin operators are now

$$S_i = \hbar \sum_{p=1}^{n} s_i^{(p)}; \quad (i = 1, 2, 3); \quad S_4 = \hbar \sum_{p=1}^{n} s_4^{(p)}. \tag{6.91}$$

It would be difficult to reproduce here the general nomenclature of spin states and (for an even number of spin 1/2 particles) the decomposition of wave functions in terms of tensor components. This nomenclature is based on the Clebsch-Gordan theorem for the product of irreducible representations, but it is completed in (de Broglie, 1934c), which defines the set of independent constants of a plane wave and the symmetry of tensors defined by an even number of particles.

These problems are treated in a different form by Fierz, whose work is based not on the fusion theory but on some conditions added to the field obeying the Klein-Gordon equation, to describe a spin $n/2$ particle. This point of view was developed by Fierz and Pauli (1939a, b) and on the basis of Dirac (1936) on the generalization of the equation of the electron, for higher spin-values.

6.4 THEORY OF PARTICLES WITH MAXIMUM SPIN 2

6.4.1 The Particles of Maximum Spin 2. Graviton

Fierz and Pauli (1939a, b) were the first to discover the connection between the equation of a particle of spin 2 and the linear approximation of the Einstein equation of a gravitation field. This approximation was given by Einstein himself (Einstein, 1916, 1918). It may be found, for instance, in Laue (1922) or Möller (1972). Einstein (1916) was the first study in which he formulated the idea of gravitational waves. He even alluded to a possible modification of gravitation theory by quantum effects, analogous to the modification of Maxwell's electromagnetism.

It must be stressed that the quantum theory of gravitation, developed by de Broglie and Tonnelat (de Broglie, 1934a,b,c; Tonnelat, 1942) on the basis of the fusion method, is not based on a particle of spin 2, but on the particle of *maximum* spin 2. This is an important point for two reasons:

1. The fusion theory raises the question: is the graviton a *composite particle*, just as the photon and all particles of spin higher than ½?
2. In the fusion theory, gravitons don't appear alone. They are linked to photons. This theory is actually a unitary theory of gravitation and electromagnetism (at least at the linear approximation), and the fields are not gathered by an extended geometry, but by the fusion of spins.

6.4.2 Why are Gravitation and Electromagnetism Linked?

When you ask why gravitation and electromagnetism are linked, formally you could say that fields are linked by Clebsch-Gordan's theorem because

$$D_{\frac{1}{2}} \times D_{\frac{1}{2}} \times D_{\frac{1}{2}} \times D_{\frac{1}{2}} = D_2 + 3D_1 + 2D_0. \tag{6.92}$$

Therefore, in the fusion of four spin 1/2 particles, we must find one particle of spin 2, three particles of spin 1, and two particles of spin 0. In particular, we have gravitons and photons. To this point we must add the spin 0 photons, the physical meaning of which is related to the Aharonov-Bohm effect, as was developed in the first part of §4.

De Broglie gave an interesting argument: he defined a particle of maximum spin 2 by the fusion of two particles of spin 1, described by the quadripotentials $A^{(1)}_\mu = \{A^{(1)}, V\}$ and $A^{(2)}_\mu = \{A^{(2)}, V\}$, and the invariants $I^{(1)}_2, I^{(2)}_2$ $(I^{(1)}_1, I^{(2)}_1 = 0$. This was because $\mu_0 \neq 0$. We are only considering the electric case. The fusion gives

$$A^{(1)}_\mu \times A^{(2)}_\mu;\;\; A^{(1)}_\mu \times I^{(2)}_2;\;\; I^{(1)}_2 \times A^{(2)}_\mu;\;\; I^{(1)}_2 \times I^{(2)}_2. \tag{6.93}$$

The first product is a tensor of rank 2 that defines a symmetric and an antisymmetric tensor:

$$A_{(\mu\nu)} = \frac{A_{(\mu\nu)} + A_{(\nu\mu)}}{2};\;\; A_{[\mu\nu]} = \frac{A_{(\mu\nu)} - A_{(\nu\mu)}}{2}. \tag{6.94}$$

The products $A_\mu^{(1)} \times I_2^{(2)}$ and $I_2^{(1)} \times A_\mu^{(2)}$ are vectorlike quantities $P_\mu^{(1)}$, $P_\mu^{(2)}$, and it may be hoped that they will be photon potentials. The antisymmetric tensor $A_{[\mu\nu]}$ suggests the electromagnetic field.

The symmetric tensor $A_{(\mu\nu)}$ cannot be interpreted at this level of exposition, but actually, we can guess that it will be related to gravitation.

De Broglie shows, owing to a study of plane waves, that $P_\mu^{(1)}, P_\mu^{(2)}$ and the antisymmetric tensor $A_{[\mu\nu]}$ are related to spin 1; $A_{(\mu\nu)}$ is linked to spin 2 only if it is reduced to a zero-spur tensor because *spur* $A_{(\mu\nu)} = A_{(\mu\mu)}$ is an invariant; and it will be actually related to spin 0, just as is the invariant $I_2^{(1)} \times I_2^{(2)}$.

Now it must be remembered that, as was shown in the case of the photon, *the splitting between different spin states is not relativistically covariant* because it is based on the total spin operator which is not a relativistic invariant. Therefore, in the fusion theory, gravitation cannot appear without electromagnetism. Furthermore, it will be shown that, if $\mu_0 \neq 0$, splitting between spin 2 and spin 0 is impossible, and the interpretation of this fact is highly significant.

6.4.3 The Tensorial Equations of a Particle of Maximum Spin 2

We give only the tensorial form generalizing § 4.1. The total wave equations [Eq. (6.11) for $n = 4$] would have $4^4 = 256$ components with 168 independent quantities (de Broglie, 1934c):

$$(\mathrm{A}) \quad \begin{array}{c} \partial_\mu \phi_{(\nu\rho)} - \partial_\nu \phi_{(\mu\rho)} = k_0 \phi_{[\mu\nu]\rho} \\ \partial_\rho \phi_{[\rho\mu]\nu} = k_0 \phi_{(\mu\nu)} \\ \partial_\mu \phi_{[\rho\sigma]\nu} - \partial_\nu \phi_{[\rho\sigma]\mu} = k_0 \phi_{[\mu\nu][\rho\sigma]} \\ \partial_\varepsilon \phi_{([\varepsilon\rho][\mu\nu])} = k_0 \phi_{[\mu\nu]\rho} \end{array}. \tag{6.95}$$

Here, $\phi_{(\mu\nu)}$ is a symmetric tensor of rank 2, $\phi_{[\mu\nu]\rho}$ is a tensor of rank 3 antisymmetric with respect to the two first indices, and $\phi_{[\mu\nu][\rho\sigma]}$ is a tensor of rank 4 antisymmetric with respect to $\mu\nu$ and $\rho\sigma$, but symmetric with respect to these pairs. A consequence of Eq. (6.95) is

$$\begin{gathered} \partial_\nu \phi_{(\mu\nu)} = \partial_\rho \partial_\nu \phi_{[\rho\mu]\nu} = 0 \\ \phi_{[\rho\rho]} = \frac{1}{2} \phi_{[\mu\rho][\mu\rho]}; \quad \partial_\nu \phi_{(\rho\rho)} = k_0 \phi_{[\nu\rho]\rho}. \end{gathered} \tag{6.96}$$

The group (B) is divided in three subgroups where new tensors of rank 2, 3, and 4 appear:

$$(\mathrm{B}_1)\quad \begin{array}{l} \partial_\mu \phi^{(1)}_{(\nu\rho)} - \partial_\nu \phi^{(1)}_{(\mu\rho)} = k_0 \phi^{(1)}_{[\mu\nu]\rho} \\ \frac{1}{2}\left(\partial_\rho \phi^{(1)}_{[\rho\mu]\nu} - \partial_\rho \phi^{(1)}_{[\rho\nu]\mu}\right) = k_0 \phi^{(1)}_{[\mu\nu]} \\ \partial_\mu \phi^{(1)}_{[\rho\sigma]\nu} - \partial_\nu \phi^{(1)}_{[\rho\sigma]\mu} = k_0 \phi^{(1)}_{[\mu\nu][\rho\sigma]} \\ \partial_\varepsilon \phi^{(1)}_{([\varepsilon\rho][\mu\nu])} = k_0 \phi^{(1)}_{[\mu\nu]\rho} \end{array}. \tag{6.97}$$

Note the antisymmetries (square brackets). From Eq. (6.97), we deduce the identities as follows:

$$\phi^{(1)}_{[\nu\mu]\nu} = \phi^{(1)}_{([\mu\nu][\rho\nu])} = 0. \tag{6.98}$$

The equations (B_2) and (B_3) are identical, and we have

$$(\mathrm{B}_2,\ \mathrm{B}_3)\quad \begin{array}{l} \partial_\mu \chi^{(1)}_\nu - \partial_\nu \chi^{(1)}_\mu = k_0 \chi^{(1)}_{[\mu\nu]} \\ \partial_\rho \chi^{(1)}_{[\rho\nu]} = k_0 \chi^{(1)}_\nu \\ \partial_\mu \chi^{(1)}_\nu = k_0 \chi^{(1)}_{\rho\nu} \\ \partial_\rho \chi^{(1)}_{[\mu\nu]} = k_0 \chi^{(1)}_{[\mu\nu]\rho} \end{array}. \tag{6.99}$$

In the third equation, $\chi^{(1)}_{\rho\nu}$ is neither symmetric nor antisymmetric. Eq. (6.99) entails

$$\begin{array}{l} \chi^{(1)}_{\rho\rho} = 0;\ \ \chi^{(1)}_{\mu\nu} - \chi^{(1)}_{\nu\mu} = \chi^{(1)}_{[\mu\nu]} \\ \chi^{(1)}_{[\nu\rho]\rho} = -\,\chi^{(1)}_\nu;\ \ \chi^{(1)}_{[\mu\nu]\rho} + \chi^{(1)}_{[\nu\rho]\mu} + \chi^{(1)}_{[\rho\mu]\nu} = 0. \end{array} \tag{6.100}$$

Finally, we find a last group of equations:

$$(\mathrm{C})\quad \begin{array}{l} \partial_\mu \phi^{(0)}_\nu = \partial_\nu \phi^{(0)}_\mu = k_0 \phi^{(0)}_{(\mu\nu)} \\ \partial_\mu \phi^{(0)}_\mu = k_0 \partial_\mu \phi^{(0)} \\ \partial_\mu \phi^{(0)} = k_0 \mu^{(0)}_\phi \end{array}. \tag{6.101}$$

The equations (B_1), (B_2), and (B_3) are three realizations of total spin 1. It is evident for (B_2) and (B_3) because putting

$$F_\mu = k_0 \chi^{(1)}_\mu;\ \ F_{[\mu\nu]} = k_0 \chi^{(1)}_{\mu\nu} \tag{6.102}$$

and defining potentials and fields as we did in Eq. (6.20), we find the Maxwell equations with mass (but we shall see that it needs some additional notation).

The correspondence is less evident for Eq. (6.97). Instead of Eq. (6.102), we must write

$$F_{\mu} = \frac{k_0}{6}\varepsilon_{\mu\lambda\nu\rho}\phi^{(1)}_{[\lambda\nu]\rho};\quad F_{[\mu\nu]} = k_0\phi^{(1)}_{\mu\nu}\ , \tag{6.103}$$

where $\varepsilon_{\mu\lambda\nu\rho}$ is the Levi-Civita symbol. Applying Eq. (6.20), we find the Maxwell equations.

Now, (C) is a realization of spin 0 as may be seen by comparing Eq. (6.101) with Eq. (6.22). But here we find a difficulty that justifies the preceding remarks: de Broglie (who did not know the magnetic case), considered only the electric photon [Eq. (6.21)] and he identified Eq. (6.101) with the non-Maxwellian equations [Eq. (6.22)]. But this implies the identity $\varphi^{(0)}= I_2$, where $\varphi^{(0)}$ is a *scalar* while I_2 is a pseudoscalar.

In de Broglie's time, people was less careful about parity than now, and he wrote that Eqs. (6.101) and (6.22) "are entirely equivalent (at least when vectors and pseudovectors are assimilated)." Today, we pay more attention to parity and we cannot neglect such a discrepancy: an equality like $\phi^{(0)} = I_2$ is unacceptable. There are two possible solutions:

1. We could admit that if $\phi^{(0)} = I_2 = 0$, $\phi^{(0)} = I_2$. Thus, the spin 0-component (C) vanishes. But there is a second spin 0-component, hidden in the equations (A) in the form of an invariant $\phi^{(0)}$, a vector $\phi^{(0)}_{\mu}$, and a symmetric tensor $\phi^{(0)}_{(\mu\nu)}$, that we can define as

$$\phi^{(0)} = \phi^{(0)}_{(\rho\rho)}\ ;\quad \phi^{(0)}_{\mu} = \phi^{(0)}_{[\mu\rho]\rho};\quad \phi^{(0)}_{(\mu\nu)} = \phi_{([\mu\rho][\nu\rho])} - \phi_{(\mu\nu)}. \tag{6.104}$$

One can show using Eq. (6.94) that these tensors obey the group C of Eq. (6.101), but once more, if $\phi^{(0)}$ is a true scalar, we can write $\phi^{(0)} = I_2$ only if $\phi^{(0)} = I_2 = 0$. This implies that Eq. (6. 101) is submitted to the condition *sp* $\phi^{(0)}_{(\rho\rho)} = 0$ that was *a priori* supposed by Fierz and Pauli, who based their theory on a spin 2 (and not a maximum spin 2) particle. De Broglie criticized this postulate as artificial.

This suggestion, based on parity, could be considered as the justification of their hypothesis. However, it may be pointed out, as de Broglie did, that the splitting of spin components is not covariant. It is, at least, the case for the condition $\phi^{(0)} = I_2 = 0$, in spite of the fact that the equality *sp* $\phi^{(0)}_{(\rho\rho)} = 0$ is covariant. Thus, the problem remains unsolved. But there is a second proposition.

2. We can ask the question: Is $\phi^{(0)} = I_2$ a good equality? Perhaps instead it is $\phi^{(0)} = I_1$, which is covariant because I_1 is a true invariant. In such a

case, Eq. (6.101) must not be identified with Eq. (6.22), but rather with Eq. (6.27). Is this possible? It seems so.

Let us go back to Eq. (6.92). The products $A_\mu^{(1)} \times I_2^{(2)}$ and $I_2^{(1)} \times A_\mu^{(2)}$, denoted as $P_\mu^{(1)}$, $P_\mu^{(2)}$, were considered by de Broglie as vectors, but he said, more prudently, that they were "vectorlike". Actually, they are pseudovectors because they are the products of a polar-vector by a pseudoscalar. Therefore, $P_\mu^{(1)}$ and $P_\mu^{(2)}$ are not polar potentials but pseudopotentials of magnetic type as are those that appear in Eq. (6.26). On the contrary, the product $I_2^{(1)} \times I_2^{(2)}$ of two pseudoscalars is a *true* scalar of the same type as I_1, which appears in Eq. (6.27), and it can be identified.

The answer to the difficulty is that the third photon associated with the graviton is not electric but magnetic.

Let us suppose that, instead of introducing only electric photons, we introduce a magnetic photon in the symbolic formulas Eq. (6.92)] with pseudopotentials $\mathbf{B}_\mu^{(1)}, \mathbf{B}_\mu^{(2)}$, and pseudoscalars $I_2^{(1)}, I_2^{(2)}$. The fusion gives

$$B_\mu^{(1)}, B_\mu^{(2)};\ \ B_\mu^{(1)} \times I_\mu^{(2)};\ \ I_\mu^{(1)} \times B_\mu^{(2)};\ \ I_\mu^{(1)} \times I_\mu^{(2)}\ , \qquad (6.105)$$

and we see the following:

- The spin 2 product: $B_\mu^{(1)} \times B_\mu^{(2)}$ has the same symmetry as $A_\mu^{(1)} \times A_\mu^{(2)}$ because the axial character of $B_\mu^{(1)} \times B_\mu^{(2)}$ is annihilated by the product.
- For the same reason, the spin 0 product $I_1^{(1)} \times I_1^{(2)}$ is a scalar, as was $I_2^{(1)} \times I_2^{(2)}$.
- The spin 1 products $B_\mu^{(1)} \times I_\mu^{(2)};\ I_\mu^{(1)} \times B_\mu^{(2)}$ are pseudovectors, as $A_\mu^{(1)} \times I_2^{(2)};\ I_2^{(1)} \times A_\mu^{(2)}$: they are products of a pseudovector by a scalar, while the latter were products of a polar vector by a pseudoscalar.

Thus, we find a magnetic photon whether we start from electric or from magnetic photons, and we can assert that one of the photons associated with the graviton is not electric but magnetic.

6.5 QUANTUM (LINEAR) THEORY GRAVITATION

6.5.1 The Particle of Maximum Spin 2. Graviton

Now, we shall follow de Broglie (1943) and Tonnelat (1942) and consider the general equations (A) when *spur* $\phi_{(\rho\rho)}^{(0)} \neq 0$. But we shall not be able to separate the spin 2 component from its spin 0 part.

We start from Eqs. (6.81) and (6.82) and the Klein-Gordon equation, verified by all the field quantities:

$$\Box\phi = -k_0^2\phi\ \ \left(\Box = -\partial_\rho\partial_\rho\right). \qquad (6.106)$$

The metric tensor $g_{(\mu\nu)}$ will be taken at the linear approximation:

$$g_{(\mu\nu)} = \delta_{\mu\nu} + h_{(\mu\nu)} \quad \left(h_{(\mu\nu)} \ll 1\right). \tag{6.107}$$

At this limit, the propagation of gravitation waves is given by

$$\Box g_{(\mu\nu)} = -2R_{(\mu\nu)} \quad \left(R_{(\mu\nu)} = g^{\rho\sigma} R_{([\mu\rho][\nu\sigma])}\right), \tag{6.108}$$

where $R_{([\mu\rho][\nu\sigma])}$ is the tensor of Riemann-Christoffel; in the Euclidian regions of space-time, we have the d'Alembert equation $\Box g_{(\mu\nu)} = 0$ without a second member. This is true if we use "isothermic" coordinates x_μ, for which $D_2 x_\mu = 0$; D_2 is the second-order Beltrami differential parameter.

Now it seems that metrics could be defined by

$$g_{(\mu\nu)} = \phi_{(\mu\nu)}. \tag{6.109}$$

But Tonnelat remarked that, according to Eq. (6.98), this implies $\partial_\mu g_{(\mu\nu)} = 0$, which is wrong because "isothermic" coordinates obey the relation[21]

$$\partial_\mu g_{(\mu\nu)} = \frac{1}{2} \partial_\nu g_{(\rho\rho)} \quad \left(g_{(\rho\rho)} = g_{(\mu\nu)} \delta^{(\mu\nu)}\right), \tag{6.110}$$

and the second member is not equal to zero. Eq. (6.96) thus contradicts Eq. (6.95), which is why Tonnelat suggested the following metrics (which is possible because $k_0 \neq 0$):

$$g_{(\mu\nu)} = \phi_{([\mu\rho][\nu\sigma])} = \phi_{(\mu\nu)} + \frac{1}{k_0^2} \partial_\mu \partial_\nu \phi_{(\rho\rho)}. \tag{6.111}$$

From this, it follows immediately that

$$\partial_\mu g_{(\mu\nu)} = \partial_\mu \phi_{([\mu\rho][\nu\sigma])} = \partial_\nu \phi_{(\rho\rho)} \tag{6.112}$$

So we get from Eqs. (6.82), (6.111), and (6.112):

$$g_{(\rho\rho)} = 2\phi_{(\rho\rho)} \rightarrow \partial_\mu g_{(\mu\nu)} = \frac{1}{2} \partial_\nu g_{(\rho\rho)} \tag{6.113}$$

in accordance with Eq. (6.96).

Now, from Eq. (6.97), we deduce that $g_{(\mu\nu)}$ obeys the Klein-Gordon equation, as other field-quantities do:

$$\Box g_{(\mu\nu)} = -k_0^2 g_{(\mu\nu)}. \tag{6.114}$$

[21] It must be noted that we do not have $g_\rho^\rho = g_{\rho\sigma} g^{\rho\sigma}$ because this quantity, *in the present case,* is equal to 4.

We have to identify Eq. (6.114) with Eq. (6.94), such that

$$R_{(\mu\nu)} = \frac{1}{2} k_0^2 g_{(\mu\nu)}. \tag{6.115}$$

Now, the tensor of Riemann-Christoffel may be deduced as the linear approximation from Eqs. (6.111), (6.81), and (6.82):

$$\phi_{([\mu\rho][\nu\sigma])} \cong \frac{2}{k_0^2} R_{([\mu\rho][\nu\sigma])}. \tag{6.116}$$

This formula is possible only if $\mu_0 \neq 0$, which imposes a curvature of the universe. Indeed, $k_0^2/2$ is nothing but the cosmological constant (which, unfortunately, Einstein disliked), defined by

$$R_{(\mu\nu)} = \lambda g_{(\mu\nu)}, \tag{6.117}$$

where λ is related to a natural curvature of space-time. In Euclidian space, $\lambda = 0$, in a de Sitter space of radius R, we have $\lambda = 3/R^2$. Therefore:

$$\lambda = \frac{k_0^2}{2} = \frac{\mu_0^2 c^2}{2\hbar^2}, \tag{6.118}$$

and the graviton mass is related to a natural curvature of radius R:

$$\mu_0 = \frac{\hbar\sqrt{6}}{Rc}. \tag{6.119}$$

If $R = 10^{26}$cm, the graviton (and photon) mass is

$$\mu_0 = 10^{-66}\text{g}. \tag{6.120}$$

The spin 0 may be eliminated from the equations of spin 2 only in one of two cases:

- By the *a priori* supposition that $\Phi^{(0)} = 0$ (Fierz equations)
- *At the limit case* $\mu_0 = 0$, when the radius of the universe is infinite—that is, the Euclidian case

In conclusion, the quantum theory of gravitation based on de Broglie's fusion theory raises the important question of a composite nature of photon and graviton, and above all, the theory furnishes the beginning of the quantum unitary field-theory of electromagnetism and gravitation. It only gives the beginning, however, because it is linear.

Two more remarks are relevant here:

- It could be asked whether the obstinate efforts of Einstein and other great physicists and mathematicians to find a unitary field theory had any basis, given that we are aware of hundreds of particles and it would seem that there is no reason to pay particular attention to only two of them: the photon and graviton. De Broglie's theory gives a reason for this, though: these particles are the only ones that are linked by spin properties in the fusion procedure. This argument is exterior to the geometrical path followed by Einstein.

- Concerning symmetry, the fact that a photon associated with the graviton could be magnetic instead of electric, as has been suggested, signifies the intrusion of duality, chirality, and magnetic monopoles instead of electric charges. *It is certainly of interest that a photon is perhaps not the one that was expected, and it must be stressed that there is another photon with zero spin.*

6.5.2 Comparison with Other Theories

First, we must emphasize the priority of Louis de Broglie in the quantum theory of the photon being considered as a *composite particle*. In his first paper on the subject (de Broglie, 1934b), the idea of a *fusion* of Dirac particles was the starting point of his theory of particles of higher spin. A second point is that, unlike others, de Broglie's initial aim was not a generalization of Dirac's equation, but a theory of light. This is why he did not introduce any electromagnetic interaction.

For these reasons, he was the only one to suppose a massive photon, unlike others who considered a massless photon as obvious. He never tried to extend his theory to massless particles and even scarcely alluded to this possibillity.

6.5.3 The "Proca Equation"

Eq. (6.21) and the very idea of a massive photon are often ascribed to Alexandru Proca. Actually, it is the result of a misunderstanding, if not a "misreading", as follows:

1. The "Proca equations" (Proca, 1936) appeared in 1936, two years after the Broglie equations (de Broglie, 1934b). Moreover, the paper of Proca was entitled: "On the ondulatory theory of positive and negative electrons". It was not a theory of the photon, but a theory of the electron: an attempt to avoid negative energies, as was frequent in that time.[22]

[22] Heisenberg and de Broglie were among the few who immediately adopted Dirac's equation, whatever the difficulties with negative energies were.

2. Rejecting spinorial wave functions, Proca suggests—for the electron—a *vectorial equation* deriving from the Lagrangian:

$$L = \frac{\hbar^2 c^2}{2} B^*_{rs} B_{rs} + m_0^2 c^4 \psi^*_r \psi_r$$
$$B_{rs} = (\partial_r - iA_r)\psi_s - (\partial_s - iA_s)\psi_r \quad (r, s = 1, 2, 3, 4). \tag{6.121}$$

The complex vectorial function ψ_r of the *electron* takes the place of de Broglie's *photon* potential $(\mathbf{A}, V)$; and A_r is a real potential of an *external* electromagnetic field acting on the electron ψ_r, and the electron—not the photon—was the object of the theory. From Eq. (6.121), Proca derived the following equations:

$$(\partial_r - iA_r)B_{rs} = k^2\psi_s, \quad (\partial_r + iA_r)B^*_{rs} = k^2\psi^*_s \quad \left(k = \frac{m_0}{\hbar c}\right), \tag{6.122}$$

and he remarked that "they have the form of Maxwell's equations [...], completed by an external potential (A_r)". But, in no way did he consider Eq. (6.122) as the equations of a photon.

Then, he gave a spin operator, but without calculating its eigenvalues and thus ignored the fact that his electron had a spin 1. This is astonishing because de Broglie worked one floor above Proca and had deduced this value 1 two years earlier in his equation of describing a massive photon (de Broglie, 1934b).

6.5.4 The Bargmann-Wigner Equation

The Bargmann-Wigner equation for higher values of spin was published in 1948 (Bargmann & Wigner, 1948) and was similar to de Broglie's equations (de Broglie, 1943). Not identical, indeed, because it lacked the idea of fusion and was restricted by an *a priori* condition of symmetry, so it had only half of the de Broglie solutions.

When the general theory is applied to the case of spin 1, Bargmann and Wigner found the equations identical with the equation taken from de Broglie's book p. 106 (de Broglie, 1943), with a difference: by virtue of their condition of symmetry, Bargmann and Wigner did not develop the wave on the 16 Clifford matrices, as we did in Eq. (6.19), but only on 10 of them: $\gamma_\mu, \gamma_{[\mu\nu]}$. The 6 others were forgotten, so the spin 1 Maxwell equations [Eq. (6.21)] were obtained, but not the non-Maxwellian ones [Eq. (6.22)], corresponding to spin 0, which clearly have an important physical meaning.

They would be unable to include the Aharonov-Bohm effect, as we did, and *a fortiori* to find the magnetic photon, of which we proved not only that it has a logical place in the theory, but that it was already hidden in de Broglie's theory and later experimentally observed. And it is the photon that automatically appears in the interaction between the leptonic monopole and the electromagnetic field.

CHAPTER 7

P, T, and C Symmetries, the Solutions with Negative Energy, and the Representation of Antiparticles in Spinor Equations

7.1 INTRODUCTION

In this chapter, we revisit the problem of P (Parity), T (Time), and C (Charge) symmetries in quantum electrodynamics, starting with the laws of Pierre Curie and classical electromagnetism rather than a priori postulating the formal covariance of quantum mechanics. It is only after having discussed these symmetries that we will assess how quantum mechanics agrees with them. In fact, it will turn out that the so-called Racah transformations are confirmed for the P and C symmetries, but not for the T symmetry. It thus evolves that there are *two* possible T transformations in the framework of classical electromagnetism, one of which is subsequently selected on the basis of a general physical argument and formal covariance. An examination of (linear) spinor equations will then reveal a large difference between the symmetries of an electric charge and those associated with a magnetic one.

It is an interesting historical point that Dirac, when first working on these questions, believed that the negative energy solutions of the Klein-Gordon equations stemmed from the fact that it is a *second*-order equation with respect to the time variable. In spite of all that he was eventually able to get from them, he was disappointed to find such solutions in his own *first*-order equation, and indeed, until the end of his life, he went on looking for an equation that would be free of such solutions. One may wonder whether these come from the *linearity* of the equation, but that is definitely not the case: negative energies, just as antiparticles are not associated with second derivatives or linearity, but rather with *symmetries* (i.e., relativistic covariance and *P, T, and C* symmetries). Recall that in special relativity, rotating bodies lead to negative *energies* [Möller (1972)], whereas in general relativity, Einstein showed, as early as 1925, that one cannot describe an electron without the appearance of a particle of opposite charge. He connected that property with *time reversal* and proved that the product *PT* reverses charges (Lochak, 1994; Einstein, 1925).

So in this chapter, our first goal will be to clarify some points pertaining to *P*, *T*, and *C* symmetries, using the work of Pierre Curie on the symmetries of the *electromagnetic field,* whose role cannot be overstated. We refrain from speaking, as some do [e.g., Berestetsky, Lifschitz, & Pitaevsky (1968)], of "charge conjugacy" about free-field equations because this leads to identifying the variance of the potentials with that of a world gradient (via gauge invariance). However, we would like to show that this variance can be *inferred* from the laws of electromagnetism. Most of this chapter is geared toward linear quantum equations; however, in Chapter 4, we already studied some *nonlinear* problems connected with chirality and the monopole, and here, we will return briefly to the nonlinear setting, with questions on the compatibility between nonlinearity and quantum mechanics. A more general and detailed study of the symmetries in nonlinear field equations can be found in Lochak (1997).

7.2 THE SPATIAL SYMMETRIES OF THE ELECTROMAGNETIC QUANTITIES

Few treatises of electromagnetism mention the *P*, *C*, and *T* symmetries. Jackson (1975) is an exception, but only formally so, as he *postulates* the invariance of the electric charge under space or time reversal[23]. We disagree about the time symmetry and, as far as space is concerned, although Jackson's choice does agree with the results of Pierre Curie, the latter does not impose them *a priori*, but rather deduces them from experimentation and his famous general symmetry law (Curie, 1894a,b):

> "Lorsque certaines causes produisent certains effets, les éléments de symétrie des causes doivent se retrouver dans les effets produits." *[When certain causes*

[23] Here is his only argument: "It is natural, convenient, and permissible to assume that charge is also a scalar under spatial inversion and even under time reversal" (Jackson, 1975, p. 249).

produce certain effects, the same patterns of symmetry should be found in the effects as in their causes.]

Conversely,

"Lorsque certains effets révèlent une certaine dissymétrie, cette dissymétrie doit se retrouver dans les causes qui lui ont donné naissance."[24] *[When certain effects betray a certain asymmetry, the latter must be present in the causes from which it originates.]*

Here are two applications of these principles, following Curie himself:

1. The spatial symmetry of the electric field: Consider an *electric* field created between two circular plates made of different metals and sharing a common axis. It displays the symmetries of the cause (i.e., of the set of the two plates): Rotation around their axes and planar symmetry with respect to any plane containing those axes, corresponding to the symmetry of a truncated cone. But there could be *more* symmetry—namely that of a cylinder[25] or a sphere. To decide this point, Curie imagines an electrically conducting sphere immersed in a uniform electric field. Then "a force will be exerted on the sphere along the direction of the field" and the asymmetry of the effect should be retraced to its cause. But the force (i.e., the effect) is not symmetric with respect to an axis orthogonal to its direction; hence, the field-sphere system admits no such axis either. Now the sphere does admit an infinity of axes of symmetry, which are included if viewed in the field because it is a conductor; so the cause of the asymmetry should lie in the field itself. The conclusion is that the electric field *exactly* admits the symmetry of a *cone* and thus can be represented as a *polar vector* of the space $\mathbb{R}^3$. The same holds true for an *electric current* or an *electric polarization*.
2. The spatial symmetry of the magnetic field: Consider now the magnetic field that is created at the center of a circular coil carrying an electric current. The axis of the coil is an *axis of isotropy* and its plane a *plane of symmetry*. So the magnetic field does possess a plane of symmetry normal to its direction. On the other hand, there is no normal axis associated with a binary symmetry. Indeed, think of a rod moving normally along its length; it does have a binary axis of symmetry spanned by the rod itself and the velocity

[24] However the effects may well be more symmetric than their causes, as some causes for asymmetry may not suffice to produce the expected effects.

[25] Curie does not elaborate, but he writes "'truncated cone'" rather than "cone," clearly because a cylinder can be seen as a particular case of the former.

vector. If one now creates a magnetic field normal to that plane, an electromotive force appears in the rod and the binary axis disappears. So it must be absent from the cause and the magnetic field admits no axis normal to its direction. The conclusion is that *the magnetic field has the symmetry of a rotating cylinder* and can be represented by an *axial vector* (in $\mathbb{R}^3$). The same can be said of a *magnetic current* or a *magnetic polarization*.

From Curie's reasonings about fields, one can deduce the symmetries of the *charges*. He does not do this himself,[26] but his reasoning can easily be adapted. Let us take up the parallel circular plates introduced here; they are swapped, as well as their charges, through a symmetry with respect to a parallel and equidistant plane. Since we know that this operation reverses the electric field, the signs of the charges do *not* change. For a magnetic charge, one can draw the opposite conclusion since in that case, the field is *not* reversed. In summary, parity reverses the sign of the *magnetic charge* (g), but not that of the *electric charge* (e):

$$P : \mathbf{E} \rightarrow -\mathbf{E};\ \ \mathbf{H} \rightarrow \mathbf{H};\ \ e \rightarrow e;\ \ g \rightarrow -g. \tag{7.1}$$

We will see in the upcoming discussion that although e is a scalar, $\mathbf{E}$ a polar vector, and $\mathbf{H}$ an axial vector, it would be wrong to conclude that the charge g is "pseudoscalar," as is sometimes asserted. In fact, *all* physical constants are *scalars,* no matter what physical quantities are characterized; no one would say that $\hbar$ is a component of an antisymmetric tensor because it is a unit of kinetic momentum. In the sequel, everything pertaining to the electric charge will remain as is, but we will have to modify our view of the magnetic charge in light of the quantum expression of chirality.

7.3 THE TIME SYMMETRY OF THE ELECTROMAGNETIC FIELD

Curie did not address the question of time reversal, which in those days was not an issue. We will first invoke Lorentz's force as exerted on an electric or magnetic charge [Jackson (1975)]:

$$\mathbf{F}_{\mathrm{elec}} = \mathrm{e}\left(\mathbf{E} + \frac{1}{\mathrm{c}}\mathbf{v} \times \mathbf{H}\right);\ \ \mathbf{F}_{\mathrm{magn}} = \mathrm{g}\left(\mathbf{H} - \frac{1}{\mathrm{c}}\mathbf{v} \times \mathbf{E}\right) \tag{7.2}$$

[26] However, he does deal with magnetic charges in his second memoir (*Curie, 1894a,b*).

Quantum mechanics, of course, should not contradict these formulas, which indeed are recovered in the semiclassical limit—especially since, although they are not enough to fix the symmetries, they should be coherent with them. Since $\mathbf{F}$ is T invariant (because $\mathbf{F} = m\mathbf{a}$) and v changes sign along with t, Eq. (7.2) forces the following transformation laws:[27]

$$T : e\mathbf{E} \to e\mathbf{E};\, e\mathbf{H} \to -e\mathbf{H};\, g\mathbf{H} \to g\mathbf{H};\, g\mathbf{E} \to -g\mathbf{E}. \tag{7.3}$$

From there, one gets *two* possible sets of laws:

$$\begin{array}{l} (I)\ \mathbf{E} \to \mathbf{E};\, \mathbf{H} \to -\mathbf{H};\, e \to e;\, g \to -g \\ (II)\ \mathbf{E} \to -\mathbf{E};\, \mathbf{H} \to \mathbf{H};\, e \to -e;\, g \to g. \end{array} \tag{7.4}$$

Hence, there is an ambiguity which is enough to lift for *one* of these quantities since the others would follow, but that seems difficult (if not impossible) with the help of electromagnetic phenomena alone. All the ones that I have tried, whether classical or quantum, may accommodate both transformation laws. This difficulty is akin to the one that Curie himself encountered concerning spatial symmetry. In his words, *"Les phénomènes généraux de l'électricité et du magnétisme nous indiquent seulement une liaison entre les symétries du champ électrique et du champ magnétique [...]"* [The general electric and magnetic phenomena tell us only about a connection between the symmetries of the electric field and those of the magnetic field.]

In fact, these phenomena only showed him that one of the fields is polar and the other axial, but without specifying which is which. And he continues: *"Pour lever cette indétermination il faut faire intervenir d'autres phénomènes, les phénomènes electrochimiques ou d'electricite de contact, les phénomènes pyro ou piézoélectriques, ou encore le phénomène de Hall, ou celui de la polarisation rotatoire magnétique."* [In order to remove this indeterminacy, one needs to bring into play other types of phenomena, such as electrochemical phenomena or those involving contact electricity, or pyro or piezoelectric ones, or the Hall phenomenon or still rotating magnetic polarization.]

In our case, too, the general electromagnetic phenomena specify only the behavior of the three remaining quantities in Eq. (7.4), provided

[27] Note that following Eq. (7.2), one can also write $P : e\mathbf{E} \to -e\mathbf{E};\, e\mathbf{H} \to e\mathbf{H};\, g\mathbf{H} \to -g\mathbf{H};\, g\mathbf{E} \to g\mathbf{E}$, and by Curie's laws for fields, one recovers Eq. (7.1) for the charges.

that one of them is known[28]. We will now follow the route suggested by Curie.

Consider an electrochemical phenomenon (e.g., cations flowing toward an anode with a density of current $\mathbf{j} = \rho \mathbf{v}$, where ρ denotes the density of cations and $\mathbf{v}$ their velocity). Let us perform a time reversal; we do not know a priori whether the sign of the charges will be reversed or not, but in either case, the signs of the ions and that of the electrode will remain opposite, so the direction of the current will remain unaffected. But since the velocity v is reversed, the charge must change sign too: so it is the second possibility in Eq. (7.4)—namely, (*II*)—that is the correct one. This result is confirmed by the reasoning of Einstein (Einstein, 1925) and based on relativistic covariance. Indeed, relativity combines the two fields **E** and **H** into an antisymmetric tensor $F_{\mu\nu}$:

$$\mathbf{E} = \{iF_{k4}\}; \mathbf{H} = \{\mathbf{F}_{kl}\} \left(x_\mu = x_k, ict\right). \tag{7.5}$$

From Eq. (7.5), Einstein concludes that the electric field, regarded as the time component of the tensor, changes sign under P and T, as does the charge density since it occurs as the divergence of the field.

When Pierre Curie investigated the symmetries of the electric field, he assumed that the charged circular plates that he considered admit an infinite number of planes of symmetry through their common axis. He thereby implicitly assumed the P invariance of the electric charge, which he had not yet demonstrated. However, he did so in the sequel, using pyroelectricity and piezoelectricity. On the other hand, he never took up the hint that he gave about electrochemical phenomena, but one may imagine that he had in mind a line of reasoning similar to the one given here. Consider, in fact, two electrodes, anode and cathode, toward which two currents, cationic and anodic, are flowing. Let us perform a symmetry that permutes the electrodes. Whether the signs of the electrodes are permuted or not, those of the ions will also permute the electrodes, and they will thus aim for the same electrode. But since these have been permuted, the ionic currents must be reversed, as do the velocities, by parity. The conclusion is that the electric charge is unchanged (i.e., it is P invariant).

[28] This is probably the reason why Jackson admits that this choice is purely conventional.

7.4 *P, T,* AND *C* VARIANCE OF THE ELECTROMAGNETIC FIELD

This analysis must be supplemented by the effect of charge conjugacy, which offers no problem in classical physics: If one reverses the sign of a charge on which a force is applied, the *external* fields are left invariant and the sign of the force [as in Eq. (7.2)] is reversed:

$$C : \mathbf{E} \to \mathbf{E}; \mathbf{H} \to \mathbf{H}; e \to -e; g \to -g \tag{7.6}$$

However, we should keep in mind that the situation will be different for fields that are *emitted* by a given charge. Following Eqs. (7.1), (7.4), and (7.6), we can write the transformation laws as

$$\begin{cases} P : \mathbf{E} - \mathbf{E}; \mathbf{H} \to \mathbf{H}; e \to e; g \to -g \\ T : \mathbf{E} \to -\mathbf{E}; \mathbf{H} \to \mathbf{H}; e \to -e; g \to g . \\ C : \mathbf{E} \to \mathbf{E}; \mathbf{H} \to \mathbf{H}; e \to -e; g \to -g \end{cases} \tag{7.7}$$

7.5 TRANSFORMING THE POTENTIALS

The *P*, *T*, and *C* transformation laws for the potentials are obtained from the actual definitions of the fields. We will simultaneously introduce the Lorentz potentials V and $\mathbf{A}$ and the pseudopotentials W and $\mathbf{B}$ associated with the magnetic monopole (Lochak, 1995a,b); note, however, that $\mathbf{B}$ has nothing to do with induction), as follows:

$$\mathbf{E} = -\nabla V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}; \mathbf{H} = curl\ \mathbf{A} \text{ and} \tag{7.8}$$

$$\mathrm{E} = curl\ \mathbf{B}; \mathbf{H} = \nabla W + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{7.9}$$

Following Eq. (7.7), one finds the following transformation laws for the potentials, which are to be satisfied by quantum equations:

$$\begin{cases} P : \mathbf{A} \to -\mathbf{A}; \mathbf{V} \to \mathbf{V}; B \to B; \mathbf{W} \to -\mathbf{W}; e \to e; g \to -g \\ T : \mathbf{A} \to \mathbf{A}; \mathbf{V} \to -\mathbf{V}; \mathrm{g} \to -B; \mathbf{W} \to \mathbf{W}; e \to -e; g \to g . \\ C : \mathbf{A} \to \mathbf{A}; \mathbf{V} \to \mathbf{V}; B \to B; \mathbf{W} \to \mathbf{W}; e \to -e; g \to -g \end{cases} \tag{7.10}$$

Lorentz transformations combine the potentials (V, $\mathbf{A}$) and (W, $\mathbf{B}$) into two 4-vectors:

$$A_\mu = (\mathbf{A}; iV); iB_\mu = (\mathbf{B}; iW). \tag{7.11}$$

According to Eq. (7.7), A_μ is a polar vector, whereas B_μ is an axial vector in space-time. In Euclidean space $\mathbb{R}^3$, **A** is a polar vector and V is a scalar vector, whereas **B** is an axial vector and W is a pseudoscalar vector[29]. These are the transformation laws that are generally agreed upon. One can check that they fit well with other results, but note that they do not discriminate between the laws of type (I) and type (II) described previously. In particular, note the following:

1. Eq. (7.10) yields the right transformation laws for Lagrangian momenta. If $\mathbf{P} = \mathbf{p} + \frac{e}{c}\mathbf{A}, E = mc^2 + eV$, or : $\mathbf{P} = \mathbf{p} + \frac{g}{c}\mathbf{B}, E = mc^2 + gW$, one gets

$$(P \text{ or } T) : \mathbf{P} \rightarrow -\mathbf{P}; \mathbf{E} \rightarrow -\mathbf{E}. \tag{7.12}$$

2. Here is a second property from pure electromagnetism. Namely, Eqs. (7.7) and (7.10), for the fields and potentials, respectively, ensure the covariance of Maxwell equations and that of de Broglie's equations for the photon (de Broglie, 1940—1942, 1943), in which potentials and fields appear on the same footing:

$$\begin{aligned} &-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = curl\ \mathbf{E}; \quad \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = curl\ \mathbf{H} + k_0^2\mathbf{A} \\ &div\ \mathbf{H} = 0; \quad div\ \mathbf{E} = -k_0^2 V \\ &\mathbf{H} = curl\ \mathbf{A}; \quad \mathbf{E} = -grad V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}; \quad \frac{1}{c}\frac{\partial V}{\partial t} + div\mathbf{A} = 0. \end{aligned} \tag{7.13}$$

This also ensures the covariance of the equations for the "magnetic photon," which involve pseudopotentials (Lochak, 1995a,b):

$$\begin{aligned} &-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = curl\ \mathbf{E} + k_0^2\mathbf{B}; \quad \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = curl\ \mathbf{H} \\ &div\ \mathbf{H} = k_0^2 W; div\ \mathbf{E} = 0 \\ &\mathbf{H} = grad W + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}; \quad \mathbf{E} = curl\ \mathbf{B}; \quad \frac{1}{c}\frac{\partial W}{\partial t} + div\mathbf{B} = 0. \end{aligned} \tag{7.14}$$

[29] Recall that a polar vector in space-time has a spatial polar component and a time component that is P invariant because that transformation acts only on the space part. On the other hand, T acts only on the time component, as is visible from Eq. (7.10). The properties of an axial vector are the opposite of that of a polar vector: see again Eq. (7.10). It may also be useful to recall that an axial vector in $\mathbb{R}^3$ is dual to a second-order antisymmetric tensor: $B_i = \frac{1}{2}\in_{ijk} C_{[jk]}$. In much the same way, an axial vector in space-time is dual to an antisymmetric tensor of third order: $B_\mu = \frac{1}{6}\in_{\mu\alpha\beta\gamma} C_{[\alpha\beta\gamma]}$.

7.6 P, T, AND C INVARIANCE IN THE DIRAC EQUATION

The Dirac equation in the presence of an electromagnetic field reads

$$\gamma_\mu\left(\partial_\mu + i\frac{e}{\hbar c}A_\mu\right)\psi + \frac{m_0 c}{\hbar}\psi = 0 \quad (x_\mu = x_k, ict) \tag{7.15}$$

where A_μ is the potential four-vector defined as in Eq. (5.4) in Chapter 5, and one sets

$$\begin{aligned} &\gamma_k = i\begin{pmatrix} 0 & s_k \\ -s_k & 0 \end{pmatrix}; k = 1,2,3; \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \\ &\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \end{aligned} \tag{7.16}$$

Here, the s_k's are the Pauli matrices:

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; s_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.17}$$

In the sequel, one should keep in mind that the vector space components and the pseudovector time components, as well as s_1, s_3, γ_2, γ_4, and γ_5, are real, whereas the vector time components and the pseudovector space components, as well as s_2, γ_1, and γ_3, are purely imaginary.

We will use Weyl's spinorial representation, which diagonalizes γ_5 and displays the chiral two- components ξ and η:

$$\psi \to U\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}; U = U^{-1} = \frac{1}{\sqrt{2}}(\gamma_4 + \gamma_5). \tag{7.18}$$

In this representation, the Dirac equation becomes (as mentioned already)

$$\begin{aligned} &\left[\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{s}.\nabla - i\frac{e}{\hbar c}(V + \mathbf{s}.\mathbf{A})\right]\xi + i\frac{m_0 c}{\hbar}\eta = 0 \\ &\left[\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{s}.\nabla - i\frac{e}{\hbar c}(V - \mathbf{s}.\mathbf{A})\right]\eta + i\frac{m_0 c}{\hbar}\xi = 0. \end{aligned} \tag{7.19}$$

Let us now write out the P, T, and C invariances of Eq. (7.15), using Eqs. (7.7) and (7.10); the P and T invariances are expressed by the Racah formulas [Racah (1937)]; note that the tilde indicates transposition:

$$\begin{aligned} &P: e \to e; x_k \to -x_k, x_4 \to x_4; A_k \to -A_k; A_4 \to A_4; \psi \to \gamma_4\psi \\ &C: e \to -e; \psi \to \gamma_2\psi^* = \gamma_2\gamma_4\tilde{\bar{\psi}}(\bar{\psi} = \psi^+\gamma_4). \end{aligned} \tag{7.20}$$

However, the Racah formula for time reversal has to be rejected because we now need to take into account the transformation formulas for the potentials and charges, which leads to writing

$$\begin{aligned} T^{(II)}_{Racah} : \; & x_k \to x_k, \; x_4 \to -x_4; A_k \to A_k; A_4 \to -A_4 \\ & \psi \to -i\gamma_1\gamma_2\gamma_3\psi; \quad e \to -e. \end{aligned} \tag{7.21}$$

This is *not* the original Racah transformation, as the latter does not modify the sign of the charge; we have added a superscript (II) to make this clear. In order to apply this transformation, one should *first* change in Eq. (7.15) the signs of time, of the charge, and of A_4, getting

$$\left\{\gamma_1\partial_1 + \gamma_2\partial_2 + \gamma_3\partial_3 - \gamma_4\partial_4 - i\frac{e}{\hbar c}(\gamma_1A_1 + \gamma_2A_2 + \gamma_3A_3 - \gamma_4A_4) + \frac{m_0c}{\hbar}\right\}\psi = 0. \tag{7.22}$$

Applying then the operator $-i\gamma_1\gamma_2\gamma_3$, we find the Dirac equation again, but with an opposite charge, so that Eq. (7.21) does not express any invariance under time reversal. This is why we will look for an antiunitary solution by taking the complex conjugate of Eq. (7.22), namely

$$\left\{-\gamma_1\partial_1 + \gamma_2\partial_2 - \gamma_3\partial_3 + \gamma_4\partial_4 - i\frac{e}{\hbar c}(\gamma_1A_1 - \gamma_2A_2 + \gamma_3A_3 - \gamma_4A_4) + \frac{m_0c}{\hbar}\right\}\psi^* = 0. \tag{7.23}$$

If we now apply the matrix $-i\gamma_3\gamma_1$, which reverses γ_1 and γ_3 while leaving γ_2 and γ_4 invariant, we retrieve Eq. (7.15) with the correct sign (i.e., a plus sign) in front of the charge. The dual matrix, $-i\gamma_4\gamma_2 = -i\gamma_5\gamma_3\gamma_1$, would result in the same signs in front of the matrices $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, but the sign of the mass would be reversed. Thus, the matrix $-i\gamma_3\gamma_1$ is the only possibility, whence the transformation laws for time reversal are as follows:

$$\begin{aligned} &T: e \to -e; x_k \to x_k, x_4 \to x_4; \\ &A_k \to A_k, A_4 \to -A_4; \psi \to -i\gamma_3\gamma_1\psi^*. \end{aligned} \tag{7.24}$$

This leads to the following P, T, and C transformations associated with the Dirac equation:

$$\begin{cases} P: e \to e; x_k \to -x_k, x_4 \to x_4; \\ A_k \to -A_k; A_4 \to A_4; \psi \to \gamma_4 \psi \\ T: e \to -e; x_k \to x_k, x_4 \to -x_4; \\ A_k \to A_k; A_4 \to -A_4; \psi \to -i\gamma_3\gamma_1\psi^* \\ C: e \to -e; \psi \to \gamma_2 \psi^* = \gamma_2\gamma_4 \tilde{\overline{\psi}} (\overline{\psi} = \psi + \gamma_4). \end{cases} \tag{7.25}$$

These transformation laws conform to the Curie laws for electromagnetism, supplemented by those for charge and time reversal that we have added. They also comply with the objections which Costa de Beauregard raised concerning the Racah transformation, invoking relativity (Costa de Beauregard, 1983). The laws given here seem clearer, and especially better-grounded, than those of Jauch and Rohrlich (1955), with whom one would also disagree somewhat. The transformation of the wave function under the T transformation [see Eqs. (7.24) and (7.25)] coincides with those in Lochak, where it appears without the charge component because the reasoning there does not take interactions into account. In fact, the T transformation proposed here is sometimes called *weak* time reversal [Sokolov and Ternov (1974)], defined as the product of charge reversal by time reversal à la Racah. For us, however, Eq. (6.10) will feature pure time reversal T, and we will consider the original Racah transformation as representing the product TC as follows:

$$T_{\text{Racah}} = TC, \tag{7.26}$$

where T_{Racah} now leaves the charge invariant:

$$\begin{aligned} &T_{\text{Racah}}: e \to e; x_k \to x_k, x_4 \to -x_4; \\ &A_k \to A_k, A_4 \to -A_4; \psi \to -i\gamma_1\gamma_2\gamma_3\psi. \end{aligned} \tag{7.27}$$

This transformation expresses a law of invariance as a product of two such. How should we interpret it? We now have *two* time-reversal operations [namely, T, as in Eq. (7.24), and T_{Racah}], and two operations that reverse the charge (namely, T and C), but with different meanings because C associates with a negative energy solution of Eq. (7.15) a positive energy solution of that same equation with the opposite charge of that of the electron. Consider, indeed, a solution with negative energy, as displayed by the minus sign in the exponential ($\omega > 0$):

$$\psi = e^{-i\omega t}\phi(\mathbf{r}). \tag{7.28}$$

The charge reversal transformation C associates with that solution one with positive energy:

$$\psi' = \gamma_2 \psi^* = e^{i\omega t} \gamma_2 \phi^* (\mathbf{r}), \tag{7.29}$$

where the plus sign comes from complex conjugacy. Now, start instead from a solution with positive energy (again $\omega > 0$):

$$\psi = e^{i\omega t} \phi(\mathbf{r}) \tag{7.30}$$

and apply the T transformation of Eq. (7.24) or (7.25). Doing this will reverse both the sign of time and of the charge, and we get a solution with opposite charge but positive energy. The inversion of the sign in the exponential due to complex conjugacy is compensated for by time reversal:

$$\psi'' = -i\gamma_3\gamma_1\psi^*(-t, \mathbf{r}) = -ie^{(-i)\omega(-t)}\gamma_3\gamma_1\phi^*(\mathbf{r}) = -ie^{i\omega t}\gamma_3\gamma_1\phi^*(\mathbf{r}). \tag{7.31}$$

Time reversal associates with an electron with positive energy a positron with an equally positive energy: the positron can be thought of as an electron going back in time, as Richard Feynman liked to put it. This would not have been the case had we adopted the first transformation law (I) in Eq. (7.4). Let us now apply the Racah transformation to the positive energy solution in Eq. (7.30):

$$\psi''' = T_{\text{Racah}}\psi = TC\psi = -i\gamma_1\gamma_2\gamma_3\psi(-t, \mathbf{r}) = -ie^{-i\omega t}\gamma_1\gamma_2\gamma_3\phi(\mathbf{r}). \tag{7.31}$$

We find again a solution going backward in time, but this is obtained via the product of time reversal [in the sense of Eq. (7.25)] and charge conjugacy, producing a minus sign in the exponential.

Finally, we can transcribe [Eq. (7.25)] in the Weyl representation [Eq. (7.18)]:

$$\begin{aligned}
&P: e \to e; \mathrm{x} \to -\mathrm{x}, t \to t; \\
&\mathbf{A} \to -\mathbf{A}, V \to V; \xi \to \eta \\
&T: e \to -e; x \to x, t \to -t; \mathbf{A} \to \mathbf{A}, V \to -V; \\
&\xi \to s_2\xi^*; \eta \to s_2\eta^* \\
&C: e \to -e; \xi \to -is_2 n^*; \eta \to is_2\xi^*.
\end{aligned} \tag{7.32}$$

It can be seen here, and it will appear more clearly later in this chapter in the case of magnetism that ξ and η are the *chiral components* of the Dirac wave function; they are permuted under parity (P) and charge conjugacy (C) but not under time reversal (T).

7.7 P, T, AND C INVARIANCE IN THE MONOPOLE EQUATION

Let us write down the linear equation for a magnetic monopole[30] (Lochak, 1983, 1984, 1985, 1987a,b, 1995a,b):

$$\gamma_\mu(\partial_\mu - g\gamma_5 B_\mu)\psi = 0. \tag{7.33}$$

The equation being massless ensures its invariance with respect to the chiral gauge transformation:

$$\psi \to \exp(i\gamma_5\theta/2)\psi;\ B_\mu \to B_\mu + \partial_\mu\theta. \tag{7.34}$$

We have seen in Eq. (7.11) that B_μ combines the pseudopotentials W and B and these appear in the Weyl representation of Eq. (7.33), to wit:

$$\begin{aligned} &\left[\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{s}.\nabla - i\frac{g}{hc}(W + \mathbf{s}.B)\right]\xi = 0 \\ &\left[\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{s}.\nabla - i\frac{g}{hc}(W - \mathbf{s}.B)\right]\eta = 0. \end{aligned} \tag{7.35}$$

This equation best displays the meaning of the Weyl representation. Indeed, comparing Eqs. (7.15) and (7.33) on the one hand with Eqs. (7.19) and (7.35) on the other makes quite clear the essential difference between an electric and a magnetic charge. In the Dirac equation [Eq. (7.15)], the charge operator is $E = eI$ (I denotes the identity matrix) with just one eigenvalue, whereas in the monopole equation [Eq. (7.33)], the operator reads $B = g\gamma_5$ with eigenvalues g and $-g$, which appear explicitly in Eq. (7.35). In fact, the main property of the Weyl representation is that it diagonalizes the charge operator B and thus separates the chiral components ξ and η:

$$UBU^{-1} = gU\gamma_5U^{-1} = g\gamma_4 = g\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{7.36}$$

As was already mentioned in §2, the pseudoscalar character of the operator B is not to be ascribed to the charge constant g but to the operator γ_5.

[30] There is no factor i in front of the charge because $B\mu$ is a pseudovector; see Eq. (7.11).

That can only be understood in quantum mechanics and is simply ignored by its classical counterpart. In fact, chirality is related to the *polarization of the wave;* the wave itself disappears in classical mechanics and its trace is reduced to a phase in the action integral.

The chiral components ξ and η corresponding to the two eigenvalues of B satisfies the two *independent* equations [Eq. (7.35)], in contrast with Eq. (7.19) for an electron, which is coupled via the mass term. The magnetic charge g occurs with opposite signs in the two equations [Eq. (7.35)] (these are the eigenvalues of the charge operator $B = g\gamma_5$), whereas the electric charge e enters with the same sign in both equations [Eq. (7.19)] (operator $E = eI$)—an essential difference between magnetism and electricity.

This entails that, at variance with what happens in classical physics, a change of sign of the magnetic charge may a priori have two different meanings in quantum mechanics:

1. It can denote a transition between two monopoles with opposite signs of their charge constants, in analogy with a transition between an electron and a positron.
2. At the same time, it can describe a transition between the two chiral components of the *same* monopole, with a given charge constant, but opposite eigenvalues of the charge operator B.

Therefore, at least in principle, we are confronted with four cases (see the first reference in Lochak, 1985):

$$\begin{aligned}
&m^+: \text{ left monopole } (\xi - \text{ component}) \text{ charge } g > 0 \\
&\overline{m}^+: \text{ right antimonopole } (\eta - \text{ component}) \text{ charge } g > 0 \\
&m^-: \text{ left monopole } (\xi - \text{ component}) \text{ charge } g < 0 \\
&\overline{m}^-: \text{ right antimonopole } (\eta - \text{ component}) \text{ charge } g < 0.
\end{aligned} \tag{7.37}$$

Let us express the P invariance, which is explicit in Eq. (7.33), changing the signs of the components of B_μ according to Eq. (7.10) but leaving the charge constant fixed g [contrary to the prescription of Eq. (7.10)—we will see why in a moment]:

$$\left\{-\gamma_1\partial_1 - \gamma_2\partial_2 - \gamma_3\partial_3 + \gamma_4\partial_4 - \frac{g}{\hbar c}(\gamma_1 B_1 + \gamma_2 B_2 + \gamma_3 B_3 - \gamma_4 B_4)\gamma_5\right\}\psi = 0. \tag{7.38}$$

Since δ_μ and B_μ transform in opposite ways and the matrices γ_4 and γ_5 anticommute, Eq. (7.38) is equivalent to Eq. (7.33) up to global multiplication by γ_4. We get the following explicit form of the P invariance for Eq. (7.33):

$$\begin{aligned} &P: g \to g; x_k \to -x_k, x_4 \to x_4; \\ &B_k \to B_k; W \to -W; \psi \to \gamma_4 \psi, \end{aligned} \tag{7.39}$$

where the magnetic charge constant is invariant which looks as if it contradicts (7.10) but this is deceptive. In fact, starting from the Weyl representation, Eqs. (7.35) and (7.39) can be rewritten as

$$\begin{aligned} &P: g \to g; x_k \to -x_k, x_4 \to x_4; \\ &B_k \to B_k; B_4 \to -B_4; \xi \leftrightarrow \eta. \end{aligned} \tag{7.40}$$

This makes it plain that whereas the charge constant is invariant, the parity operator swaps the chiral components ξ and η, hence the eigenvalues $\pm$g of the charge operator B. It thus changes the sign of the charge of the monopole, as predicted in Eq. (7.10), although this is effected not via changing the sign of the charge constant but rather a change of chirality, in conformity with Curie's memoir (see §7.2). Following the terminology in Eq. (7.37), we will get, according to the sign of g (for a priori, both values $+g$ and $-g$ could exist in nature):

$$P: m^{+} \leftrightarrow \overline{m}^{+} \text{ or } m^{-} \leftrightarrow \overline{m}^{-}. \tag{7.41}$$

Let us now take a look at the charge conjugation operator C. To this end, we take the complex conjugate of Eq. (7.33), keeping Eqs. (7.11) and (7.16) in mind:

$$\left\{-\gamma_1\partial_1 + \gamma_2\partial_2 - \gamma_3\partial_3 - \gamma_4\partial_4 - \frac{g}{\hbar c}(\gamma_1 B_1 - \gamma_2 B_2 + \gamma_3 B_3 + \gamma_4 B_4)\gamma_5\right\}\psi^* = 0. \tag{7.42}$$

Multiplying by γ_2, we fall back onto Eq. (7.33), which is thus invariant under C, but again *without* changing the sign of the constant g:

$$C: g \to g; \psi \to \gamma_2\psi^* = \gamma_2\gamma_4\tilde{\overline{\psi}} \ \left(\overline{\psi} = \psi^{+}\gamma_4\right). \tag{7.44}$$

Here again, this may seem to contradict Eq. (7.10) but using the Weyl representation, Eq. (7.43) reads

$$C: g \to g; \xi \to -is_2\eta^*; \eta \to -is_2\xi^*. \tag{7.44}$$

In other words, charge conjugacy leaves Eq. (7.33) invariant, as well as the equivalent *system* of equations [Eq. (7.35)] without changing the sign of the constant g; but inside Eq. (7.35), it leads to permuting the chiral components; that is, the left and right monopoles with respective eigenvalues $\pm g$ of the charge operator. The charge changes sign much as in Eq. (7.41):

$$C : m^+ \leftrightarrow \overline{m}^+ \ or \ m^- \leftrightarrow \overline{m}^-. \tag{7.45}$$

Finally we come to the time reversal transformation T, by introducing Eq. (7.10) into Eq. (7.33):

$$\left\{\gamma_1\partial_1 + \gamma_2\partial_2 + \gamma_3\partial_3 - \gamma_4\partial_4 - \frac{g}{\hbar c}(-\gamma_1 B_1 - \gamma_2 B_2 - \gamma_3 B_3 + \gamma_4 B_4)\gamma_5\right\}\psi = 0. \tag{7.46}$$

Apply -$i\gamma_1\gamma_2\gamma_3$i, which commutes with γ_1,γ_2,γ_3, and anticommutes with γ_4 et γ_5 to retrieve Eq. (7.33) again and a Racah transformation, which looks compatible with Eq. (7.10):

$$\begin{aligned} T_{\text{Racah}} &: g \to g; x_k \to x_k, x_4 \to -x_4; \\ B_k &\to -B_k, B_4 \to B_4; \psi \to -i\gamma_1\gamma_2\gamma_3\psi. \end{aligned} \tag{7.47}$$

We check what happens to the chiral components by transcribing Eq. (7.47) in the Weyl representation:

$$\begin{aligned} T_{\text{Racah}} &: g \to g; x_k \to x_k, x_4 \to -x_4; \\ B_k &\to -B_k, B_4 \to B_4 \\ \xi &\to -i\eta; \eta \to i\xi. \end{aligned} \tag{7.48}$$

One finds that the charge constant is left invariant but that again chirality is not, permuting the ξ and η components, hence the charge of the monopole, which violates Eq. (7.10), which we took as our starting point. So that Racah transformation is *not* admissible for representing time reversal for a magnetic charge. This conclusion contradicts my own previous papers on the subject, where I had overlooked (and so apparently did everyone else) the P, T, and C transformation laws for electromagnetic quantities, as embodied by Eq. (7.10). On the other hand, the unitary transform of §7.6 still holds true here. Indeed, consider the complex conjugate of Eq. (7.46), taking Eq. (7.10) into account:

$$\left\{-\gamma_1\partial_1 + \gamma_2\partial_2 + \gamma_3\partial_3 - \gamma_4\partial_4 + \frac{g}{\hbar c}(\gamma_1 B_1 - \gamma_2 B_2 + \gamma_3 B_3 - \gamma_4 B_4)\gamma_5\right\}\psi^* = 0. \tag{7.49}$$

Multiplying by $-i\gamma_3\gamma_1$ changes the sign of γ_1 and γ_3, leaving γ_2, γ_4, and γ_5 invariant and it takes us back to Eq. (7.33), leading to the following transformation law:

$$\begin{aligned} T &: g \to g; x_k \to x_k, x_4 \to -x_4; \\ B_k &\to -B_k, B_4 \to B_4; \psi \to -i\gamma_1\gamma_2\gamma_3\psi^*, \end{aligned} \tag{7 50}$$

but now in the Weyl representation, we find:

$$\begin{aligned} &T: g \to g; x_k \to x_k, t \to -t; \\ &B_k \to -B_k, W \to W; \\ &\xi \to -s_2\xi^*; \eta \to s_2\eta^*, \end{aligned} \tag{7.51}$$

which makes it clear that, at variance with the Racah transformation [Eq. (7.48)], the chiral components are *not* permuted and the magnetic charge stays invariant, in conformity with our starting point [Eq. (7.10)]. Summarizing, and in parallel with the transformation laws [Eq. (7.25)] for the electron, we arrive at the following table for the monopole:

$$\begin{aligned} &\mathrm{P}: g \to g; x_k \to -x_k, x_4 \to x_4; \\ &B_k \to B_k; B_4 \to -B_4; \psi \to \gamma_4\psi \\ &\mathrm{T}: g \to g; x_k \to x_k, x_4 \to -x_4; \\ &B_k \to B_k; B_4 \to B_4; \psi \to -i\gamma_3\gamma_1\psi^* \\ &C: g \to g; \psi \to \gamma_2\psi^* = \gamma_2\gamma_4\tilde{\bar{\psi}}\left(\bar{\psi} = \psi^+\gamma_4\right), \end{aligned} \tag{7.52}$$

which can be rewritten in the Weyl representation as

$$\begin{aligned} &P: g \to g; x_k \to -x_k, t \to t; \\ &B_k \to B_k; W \to -W; \xi \leftrightarrow \eta \\ &T: g \to g; x_k \to x_k, t \to -t; \\ &B_k \to -B_k; W \to W; \xi \to s_2\xi^*; \eta \to s_2\eta^* \\ &C: g \to g; \xi \to -is_2\eta^*; \eta \to is_2\xi^*. \end{aligned} \tag{7.53}$$

As was mentioned in §7.6 already, chirality is seen much more clearly with magnetism than it is with electricity. The system [Eq. (7.35)] is made of two independent equations for ξ and η, respectively, and formulas [Eq. (7.53)] show that they are exchanged under parity and under charge conjugacy. There are two monopoles, left and right, which form a particle-antiparticle pair. Eq. (7.35) also displays the neutrino as a particular case or, conversely, suggests that monopoles can be considered as "magnetically excited" neutrinos. This leads to a question that we have raised previously: Could it be that such monopoles are produced in weak interactions, in which case the fact that they interact strongly with matter, in contrast with neutrinos, could explain the deficit of solar neutrinos in terrestrial observations?

Let us stress one more time that in Eq. (7.53), the P and C transformations leave the magnetic charge constant *invariant,* which does not contradict Eq. (7.10) because *the charge of a monopole changes via its chirality*. Chirality remains invariant under time reversal, and thereby so does the magnetic charge, contrary to what happens with the electric charge. In a figurative way, and in sharp contrast with the case of the electron, one can say that *an antimonopole is not a monopole going back in time, but rather its mirror image*. It thus seems that one should consider a monopole with a charge constant $-g$ not as the antiparticle partner of one of charge $+g$, but rather as another particle altogether.

7.8 *P, T,* AND *C* TRANSFORMATION LAWS FOR TENSOR QUANTITIES

Consider the 16 tensor quantities associated with the Dirac equation:

$$\begin{aligned}&\omega_1 = \overline{\psi}\psi; J_\mu = i\overline{\psi}\gamma_\mu\psi; M_{\mu\nu} = -i\overline{\psi}\gamma_\mu\gamma_\nu\psi;\\&\Sigma_\mu = -i\overline{\psi}\gamma_\mu\gamma_5\psi; \omega_2 = -i\overline{\psi}\gamma_5\psi,\end{aligned} \tag{7.54}$$

or in the Weyl representation:

$$\begin{aligned}&\omega_1 = \xi^+\eta + \eta^+\xi; \omega_2 = i\left(\xi^+\eta - \eta^+\xi\right)\\&\omega_1^2 + \omega_2^2 = 4\left(\xi^+\eta\right)\left(\eta^+\xi\right)\\&J_\mu = \{J_4, \mathbf{J}\} = \left\{\mathrm{i}\left(\xi^+\xi + \eta^+\eta\right), -\left(\xi^+\mathrm{s}\xi - \eta^+\mathrm{s}\eta\right)\right.\\&\Sigma_\mu = \{\Sigma_4, \boldsymbol{\Sigma}\} = \left\{i\left(\xi^+\xi - \eta^+\eta\right), -\left(\xi^+\mathrm{s}\xi + \eta^+\mathrm{s}\eta\right)\right\}\\&M_{\mu\nu} = \left\{M_{j4}, M_{jk}\right\} = \left\{\xi^+\mathrm{s}\eta - \eta^+\mathrm{s}\xi\right), \left(\xi^+\mathrm{s}\eta + \eta^+\mathrm{s}\xi\right)\right\}.\end{aligned} \tag{7.55}$$

We know that ω_1 and ω_2 are Lorentz invariants, J_μ and S_μ are vectors, and $M_{\mu\nu}$ is an antisymmetric tensor. Their behaviors under P, T, and C are displayed in Table 7.1, together with that of the *chiral currents* (Lochak, 1959, 1983), which are consequences of Eq. (7.55):

$$X_\mu = \left\{i\xi^+\xi, -\xi^+\mathrm{s}\xi\right\}; Y_\mu = \left\{i\eta^+\eta, \eta^+\mathrm{s}\eta\right\}. \tag{7.56}$$

From Eq. (7.55), it is plain that

$$J_\mu = X_\mu + Y_\mu; \Sigma_\mu = X_\mu - Y_\mu. \tag{7.57}$$

Now there is a table of the P, T, and C transformation laws as derived from Eqs. (7.25), (7.32), (7.52), and (7.53), and in accordance with (*II*) in Eq. (7.4):

Let us now introduce the physical dimensions of these quantities, taking into account the P, T, and C transformation rules for the charges. We denote by $\mathbf{P} = -\frac{e\hbar}{2m_0c}M_{4k}$ and $\mathbf{M} = -\frac{e\hbar}{2m_0c}M_{kl}$ the electric and magnetic polarizations of the electron, respectively. The transformation rules for the electric charge are taken from Eq. (7.10), but those for the magnetic charge have been modified according to quantum mechanics, as in Eqs. (7.52) and

Table 7.1

		P	T	C
ψ	$\rightarrow$	$\gamma_4\psi$	$-i\gamma_3\gamma_1\psi^*$	$\gamma_2\psi^*$
ξ	$\rightarrow$	η	$s_2\xi^*$	$-is_2\eta^*$
η	$\rightarrow$	ξ	$s_2\eta^*$	$is_2\eta^*$
ω_1	$\rightarrow$	ω_1	ω_1	$-\omega_1$
J_4	$\rightarrow$	J_4	J_4	J_4
J_k	$\rightarrow$	$-J_k$	$-J_k$	J_k
M_{4k}	$\rightarrow$	$-M_{4k}$	M_{4k}	M_{4k}
M_{jk}	$\rightarrow$	M_{jk}	$-M_{jk}$	M_{jk}
$\sum_4$	$\rightarrow$	$-\sum_4$	$\sum_4$	$-\sum_4$
$\sum_\kappa$	$\rightarrow$	$\sum_\kappa$	$-\sum_\kappa$	$-\sum_\kappa$
ω_2	$\rightarrow$	$-\omega_2$	$-\omega_2$	$-\omega_2$
$\xi^+\xi$	$\rightarrow$	$\eta^+\eta$	$\xi^+\xi$	$\eta^+\eta$
$\xi^+s_\kappa\xi$	$\rightarrow$	$\eta^+s_\kappa\eta$	$-\xi^+s_\kappa\xi$	$-\eta^+s_\kappa\eta$
$\eta^+\eta$	$\rightarrow$	$\xi^+\xi$	$\eta^+\eta$	$\xi^+\xi$
$\eta^+s_\kappa\eta$	$\rightarrow$	$\xi^+s_\kappa\xi$	$-\eta^+s_\kappa\eta$	$-\xi^+s_\kappa\xi$

(7.53). The sign changes do not come from the constant g, but from the changes of chirality.

We will now give the transformation laws for the fields and the potentials, but at variance with Eqs. (7.7) and (7.10), the fields are not external anymore; rather, they are caused by the currents. The P and T rules are identical, but this is not so for C. In order to check the rules for parity and time reversal, it is enough to check that the rules previously noted for the fields are in accordance with those for the currents. But for charge conjugacy, the currents (i.e., the cause) determine the rule for the fields (i.e., the effect) via the covariance that Curie's laws imply for Maxwell equations (as discussed earlier in this chapter); we thus add the subscript *em* (meaning "emitted") to the fields in the table:

$$A_\mu = \frac{4\pi}{c} eJ_\mu;\ B_\mu = \frac{4\pi}{c} g\Sigma_\mu, \tag{7.58}$$

or, more explicitly:

$$\begin{aligned} V &= \frac{4\pi}{c} eJ_4;\ \mathbf{A}_\mu = \frac{4\pi}{c}\mathbf{J}; \\ W &= \frac{4\pi}{c} g\Sigma_4;\ \mathbf{B} = \frac{4\pi}{c} g\mathbf{\Sigma}. \end{aligned} \tag{7.59}$$

So here is the revised version of the table, given as Table 7.2:

Table 7.2

		P	T	C
e	$\rightarrow$	e	$-e$	$-e$
g	$\rightarrow$	g	g	g
eJ_4	$\rightarrow$	eJ_4	$-eJ_4$	$-eJ_4$
$e\mathbf{J}$	$\rightarrow$	$-e\mathbf{J}$	$e\mathbf{J}$	$-e\mathbf{J}$
$\mathbf{P}$	$\rightarrow$	$-\mathbf{P}$	$-\mathbf{P}$	$-\mathbf{P}$
$\mathbf{M}$	$\rightarrow$	$\mathbf{M}$	$\mathbf{M}$	$-\mathbf{M}$
$g\sum_4$	$\rightarrow$	$-g\sum_4$	$g\sum_4$	$-g\sum_4$
$g\sum$	$\rightarrow$	$g\sum$	$-g\sum$	$-g\sum$
$\mathbf{E}$	$\rightarrow$	$-\mathbf{E}$	$-\mathbf{E}$	$-\mathbf{E}_{(em)}$
$\mathbf{H}$	$\rightarrow$	$\mathbf{H}$	$\mathbf{H}$	$-\mathbf{H}_{(em)}$
V	$\rightarrow$	V	$-V$	$-V_{(em)}$
$\mathbf{A}$	$\rightarrow$	$-\mathbf{A}$	$\mathbf{A}$	$-\mathbf{A}_{(em)}$
W	$\rightarrow$	$-W$	W	$W_{(em)}$
$\mathbf{B}$	$\rightarrow$	$\mathbf{B}$	$-\mathbf{B}$	$\mathbf{B}_{(em)}$

By using this table, one can check easily the covariance of the polarizations and fields, as well as that of the currents and potentials; it makes plain that the correct transformation rules hold true for the quantities with physically meaningful coefficients. Notice also that the T invariance of $e\mathbf{J}$ in Table 7.2, which results from Eq. (7.24) goes along with the discussion in §7.3 leading to the choice of the second possibility (*II*) in Eq. (7.4).

Finally, Table 7.1 provides a new argument in favor of the T transformation and against the original Racah transformation. Indeed, one finds from the table that the first invariant ω_1 is a true invariant in space-time, being both P and T invariant, whereas ω_2 appears as a *pseudo*invariant, changing sign under P and T. The T invariance of ω_1 is especially important, as it ensures the invariance of the Dirac Lagrangian equation:

$$L = \hbar c\left\{\overline{\psi}\gamma_\mu\left(\frac{1}{2}\left[\partial_\mu\right] + i\frac{e}{\hbar c}A_\mu\right)\psi + \frac{m_0 c}{\hbar}\overline{\psi}\psi\right\}$$
$$\left\{\left[\partial_\mu\right] = \left(\partial_\mu\rightarrow\right) - \left(\leftarrow\partial_\mu\right)\right\}, \tag{7.60}$$

and thereby the T invariance of the energy density:

$$E = \frac{\partial L}{\partial\left(\frac{\partial\psi}{\partial t}\right)}\left(\frac{\partial\psi}{\partial t}\right) + \left(\frac{\partial\overline{\psi}}{\partial t}\right)\frac{\partial L}{\partial\left(\frac{\partial\overline{\psi}}{\partial t}\right)} - L. \tag{7.61}$$

By contrast, using the Racah version of the T transformation, we would get

$$T_{\text{Racah}} : \omega_1 \rightarrow -\omega_1 \;\; \text{hence} : E \rightarrow -E, \tag{7.62}$$

and this property suffices to rule out this transformation law; indeed, recall that an energy density appears as the T_{44}-component of the energy-momentum tensor and varies as the square of the time variable, which immediately entails its T invariance.

7.9 NONLINEARITY AND QUANTUM MECHANICS: ARE THEY COMPATIBLE?

We start from a question that seems to be simple enough: What are the main features of the Dirac equation that ensure that it complies with the general principles of quantum mechanics, and in particular, produces the correct semiclassical approximation? Recall the original Dirac equation [Eq. (7.15)] and its Weyl representation [Eq. (7.19)], which diagonalize the matrix γ_5, displaying the chiral components. In order to derive Planck's law from the Dirac equation, it is best to write the corresponding Lagrangian equation as follows:

$$\begin{aligned} L &= \hbar c\left\{\overline{\psi}\gamma_\mu\left(\frac{1}{2}[\partial_t] + i\frac{e}{\hbar c}A_\mu\right)\psi + \kappa_0\overline{\psi}\psi\right\}; \;\; \{[\partial] = (\partial\rightarrow) - (\leftarrow\partial)\} \\ &= \frac{\hbar c}{i}\left\{\xi^+ + \left(\frac{1}{2c}[\partial_t] - \frac{ie}{\hbar c}V\right)\xi - \xi^+ s.\left(\frac{1}{2}[\nabla] + \frac{ie}{\hbar c}\mathbf{A}\right)\xi\right. \\ &\quad \left. + \eta^+\left(\frac{1}{2c}[\partial_t] - \frac{ie}{\hbar c}V\right)\eta + \eta^+ s\left(\frac{1}{2}[\nabla] + \frac{ie}{\hbar c}\mathbf{A}\right)\eta + i\kappa_0\left(\xi^+\eta + \eta^+\xi\right)\right\}, \end{aligned} \tag{7.63}$$

with $\kappa_0 = \frac{m_0 c}{\hbar}$. This determines in turn the energy-momentum tensor, and thus the following energy density:

$$\begin{aligned} E &= \frac{\partial L}{\partial(\partial_t\psi)}(\partial_t\psi) + \left(\partial_t\overline{\psi}\right)\frac{\partial L}{\partial\left(\partial_t\overline{\psi}\right)} - L \\ &= \frac{\partial L}{\partial(\partial_t\xi)}(\partial_t\xi) + \left(\partial_t\xi^+\right)\frac{\partial L}{\partial\left(\partial_t\xi^+\right)} + \frac{\partial L}{\partial(\partial_t\eta)}(\partial_t\eta) + \left(\partial_t\eta^+\right)\frac{\partial L}{\partial(\partial_t\eta^+)} - L. \end{aligned} \tag{7.64}$$

Since the equations of the motion imply that L vanishes (a point to which we will return later in this discussion), this density can be rewritten as

$$E = \frac{\hbar}{2i}\left(\overline{\psi}\partial_t\psi - \partial_t\overline{\psi}\psi\right) = \frac{\hbar}{2i}\left\{\left(\xi^+\partial_t\xi - \partial_t\xi^+\xi\right) + \left(\eta^+\partial_t\eta - \partial_t\eta^+\eta\right)\right\}. \tag{7.65}$$

Consider a stationary wave

$$\psi = e^{i\omega t}\phi(\mathbf{r}), \tag{7.66}$$

with energy density

$$E = \phi^+\phi\ \hbar\omega = \left(\xi^+\xi + \eta^+\eta\right)\hbar\omega. \tag{7.67}$$

Assuming that the wave function is normalized, we find after integrating over the whole space,

$$E = \hbar\omega, \tag{7.68}$$

confirming that Planck's laws follow from the Dirac equation. Let us now compute the plane waves in the Weyl representation [Eq. (7.19)] with no external field; to this end, set

$$\xi = ae^{i(\omega t - \mathrm{k.r})};\ \ \eta = be^{i(\omega t - \mathrm{k.r})}, \tag{7.69}$$

in which ω, $\mathbf{k}$, a, and b are constants (a and b being spinors). We then get

$$\begin{aligned}\left(\frac{\omega}{c} + \mathbf{s.k}\right)a + k_0 b = 0\\ \left(\frac{\omega}{c} - \mathbf{s.k}\right)a + k_0 b = 0.\end{aligned} \tag{7.70}$$

Setting the determinant equal to 0 yields the dispersion relation in the following form:

$$\left(\frac{\omega}{c}\right)^2 = k^2 + k_0^2;\ \ \omega = 2\pi\nu;\ \ \mathbf{k} = \left(\frac{2\pi}{\lambda}\right)\mathbf{n}, \tag{7.71}$$

with frequency ν and wavelength λ (with $\mathbf{n}$ as a unit vector). Multiplying the phase in Eq. (7.69) by Planck's constant and using Eq. (7.68), we find

$$\hbar\omega t - \hbar\mathbf{k.r} = Et - \left(\frac{h}{\lambda}\right)\mathbf{n.r}. \tag{7.72}$$

However, by definition, plane waves belong to the domain of geometrical optics, and the connection with quantum mechanics is effected by identifying this phase with the action integral of Hamiltonian classical mechanics. In other words, $\hbar\mathbf{k}$ represents a Lagrange momentum $\mathbf{p}$, which immediately yields de Broglie's wavelength formula:

$$\lambda = \frac{h}{p}. \tag{7.73}$$

This formula is thus included, along with Planck's law, in the Dirac equation. Finally, let us multiply the dispersion relation [Eq. (7.71)] by $\hbar^2$, taking Eq. (7.73) into account. We retrieve the expression of the energy E, whence come the Hamilton-Jacobi equation and the semiclassical approximation, expressing E and $\mathbf{p}$ as the space-time gradient of the action:

$$\left(\frac{E}{c}\right)^2 = \mathbf{p}^2 + m_0^2c^2. \tag{7.74}$$

At this point, these results depend heavily on the *linearity* of the Dirac equation. In particular, in order to get Planck's law, we used the vanishing of the Lagrangian expression in Eq. (7.64), which is a consequence of linearity; or, equivalently, that this Lagrangian is quadratic. Besides, one then needs to integrate Eq. (7.71) over space, which requires the wave function to be normalized—another consequence of linearity.

One could be led to think that linearity is not really necessary and that one could accommodate a first-degree homogeneous equation, which makes it possible to normalize the wave function and entails the vanishing of the Lagrangian along any solution. However, this is not sufficient to recover de Broglie's wavelength formula. Indeed, to make sense of Eq. (7.73) and recover the Hamilton-Jacobi equation, it is first necessary that Eq. (7.71) give the general dispersion relation and that it be identified (to within a term of order h^2) with the expression [Eq. (7.74)] of the energy. But Eq. (7.74) is imposed by relativistic considerations, and Eq. (7.71) agrees with it for the Dirac equation because iterating the latter leads to the Klein-Gordon equation, which was built out of Eq. (7.74). Note that this is not a fortuitous coincidence—rather, it was built in from the very beginning. It is thus clear that this will not occur for a nonlinear equation, if only because the dispersion relation will be different: it may happen that some solutions are admissible in the semiclassical approximation, but that will almost never be the case for the general solution.

Besides, being in accordance with quantum mechanics requires more than Planck's law and de Broglie's formula. One still needs the so-called superposition principle, which by definition will not hold true in the nonlinear case, except asymptotically in regions with vanishingly small wave function and a weak nonlinearity. This is what de Broglie was counting on in his theory of what he called "*onde à bosse*" (one-bump wave), whose goal was to describe the link between the wave and the particle by representing the latter as an intense region of the wave.

We are thus led to the conclusion that in general, when looking for *nonlinear* wave equations, one is drifting away from quantum mechanics, even in the semiclassical approximation. It thus seems naive to look for a nonlinear version of quantum mechanics that could replace and improve the one we are used to. Nonlinear equations tell another story altogether, and their connection with quantum or even classical mechanics can be at best asymptotic. The search for such equations can thus be meaningful only inasmuch as one is determined to make a foray into unchartered territories where quantum mechanics itself does not venture, such as the structure of particles, the connection between waves and particles, the description of quantum transitions, etc.

7.10 NONLINEAR SPINORIAL EQUATIONS AND THEIR SYMMETRIES

Now we are going to partially extend to nonlinear equations the results of this chapter thus far, confining our attention to relativistic Lagrangian equations in which only the mass term is nonlinear while a linear differential part is retained. We start with the general form of the Lagrangian equation in the electric case:

$$\begin{aligned} L_e &= L_D + \hbar c F(\omega_1, \omega_1) \\ &= \hbar c \left\{ \overline{\psi} \gamma_\mu \left(\frac{1}{2} \left[\partial_\mu \right] + i \frac{e}{\hbar c} A_\mu \right) \psi + F(\omega_1, \omega_2) \right\}, \end{aligned} \tag{7.75}$$

in which L_D is the differential part of Dirac's Lagrangian and $F(w_1, W_2)$ is an as-yet-arbitrary function, with dimension L^{-1} being the inverse of a length. We insist that this nonlinear term is the most general one possible and that it subsumes all terms of the following form:

$$F(J_\mu J_\mu);\ \ F\Big(\sum_\mu \sum_\mu\Big);\ \ F\big(M_{\mu\nu}\tilde{M}_{\mu\nu}\big);\ \ F\Big(J_\mu \sum_\nu M_{\mu\nu}\Big)\text{etc.} \tag{7.76}$$

We now define *chiral invariance* via the magnetic gauge transform, which we have already discussed at length, namely

$$\psi \rightarrow e^{i\gamma_5\theta/2}\psi \ \ (\theta = \text{Const.}). \tag{7.77}$$

Now recall the Weyl transform diagonalizing the matrix γ_5 and splitting Eq. (7.77) into two transformations that exchanged the chiral components (ξ, η) of ψ:

$$\xi \rightarrow e^{i\theta/2}\xi; \quad \eta \rightarrow e^{-i\theta/2}\eta. \tag{7.78}$$

Performing this transformation leaves the tensor quantities X_μ, Y_μ, J_μ, and $\sum_\mu$ invariant since they do not contain mixed ξ-η terms. But the ω_i's $(i = 1, 2)$ do, and that gauge transformation indeed induces a rotation of the (ω_1, ω_2)-plane by an angle θ. Namely, we have (Lochak, 1983):

$$\begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{7.79}$$

This plane is chiral because whereas ω_1 is a relativistic invariant, ω_2 is pseudo-invariant such that the sense of rotation changes with parity, and θ is also a pseudo-invariant, in contrast with the phase angle of the electron. This is a fundamental difference between electricity and magnetism, with which Maxwell and Pierre Curie were fully familiar in the framework of classical physics.

Define the polar coordinates as ρ and the angle A (not to be confused, of course, with the Lorentz potential); here again, ρ is invariant and A is pseudo-invariant:

$$\rho = \sqrt{\omega_1^2 + \omega_2^2} = 2\sqrt{(\xi + \eta)(\eta + \xi)}; A = \arctan\frac{\omega_2}{\omega_1} \tag{7.80}$$

The transformation [Eq. (7.78)] now amounts to the rotation $A \rightarrow A + \theta$, and we see that the chiral-invariant equations are derived from a Lagrangian expression that is invariant under rotations of the chiral plane: the nonlinear term depends on ρ only, not on A:

$$F(\omega_1, \omega_2) = F\left(\sqrt{\omega_1^2 + \omega_2^2}\right) = F\left(2\sqrt{(\xi + \eta)(\eta + \xi)}\right) = F(\rho). \tag{7.81}$$

Now recall the linear equations [Eqs. (7.33) and (7.35)] for the magnetic monopole. We thus find that the general nonlinear chiral-invariant Lagrangian expression in the Weyl representation reads as follows (Lochak, 1983, 1984, 1985, 1987a,b):

$$
\begin{aligned}
L_m &= \hbar c\left[\overline{\psi}\gamma_\mu\left(\frac{1}{2}\left[\partial_\mu\right] - \frac{g}{\hbar c}\gamma_5 B_\mu\right)\psi + F(\rho)\right] \\
&= \frac{\hbar c}{i}\left\{\xi^+\left(\frac{1}{2}\frac{1}{c}[\partial_t] - \frac{g}{\hbar c}W\right)\xi - \xi^+ s.\left(\frac{1}{2}[\nabla] + \frac{g}{\hbar c}\mathbf{B}\right)\xi\right. \\
&\quad \left. + \eta^+\left(\frac{1}{2}\frac{1}{c}[\partial_t] + \frac{g}{\hbar c}W\right)\eta + \eta^+ s.\left(\frac{1}{2}[\nabla] - \frac{g}{\hbar c}\mathbf{B}\right)\eta + iF(\rho)\right\},
\end{aligned}
\tag{7.82}
$$

with ρ as in Eq. (7.80) and an arbitrary function F. From this Lagrangian, one first derives in the Dirac representation the following equation:

$$
\gamma_\mu\left(\partial_\mu - \frac{g}{\hbar c}\gamma_5 B_\mu\right)\psi + k(\rho)\frac{\omega_1 - i\gamma_5\omega_2}{\sqrt{\omega_1^2 + \omega_2^2}}\psi = 0, \tag{7.83}
$$

where k is the derivative of F: $k = F'(\rho)$. In the Weyl representation, this reads as

$$
\begin{aligned}
&\frac{1}{c}\frac{\partial\xi}{\partial t} - \mathbf{s}.\nabla\xi - i\frac{g}{\hbar c}(W + \mathbf{s}.\mathbf{B})\xi + ik(\rho)\sqrt{\frac{\eta + \xi}{\xi + \eta}}\eta = 0 \\
&\frac{1}{c}\frac{\partial\eta}{\partial t} + \mathbf{s}.\nabla\eta + i\frac{g}{\hbar c}(W - \mathbf{s}.\mathbf{B})\eta + ik(\rho)\sqrt{\frac{\xi + \eta}{\eta + \xi}}\xi = 0.
\end{aligned}
\tag{7.84}
$$

So these are the most general possible equations for a magnetic monopole. Note that the mass term of each of the equations in Eq. (7.84) has the same phase as the corresponding differential term, indicating a phase independence between the chiral components ξ and η, which is absent from the Dirac equation. This extra degree of freedom comes from the chiral invariance, which the Dirac equation does not possess, and this is a fundamental difference. Eq. (7.84) leaves magnetism invariant, whereas the Dirac equation does not, at least in general. Only a subset of its solutions enjoys this property—namely, those that satisfy the so-called Majorana condition (Lochak, 1992). These solutions live on the light cone and leave the norms of the isotropic currents X_μ and Y_μ invariant.

On the other hand, if one introduces a term representing magnetic interaction in the Dirac equation, one does not get an equation for a magnetic monopole (except in special cases). And if one replaces the magnetic interaction by an electric one in an equation of the same type as Eq. (7.83), the result does not represent an electron [Daviau (2005), Daviau & Lochak (1991)].

We will now examine two particular cases of Eqs. (7.83) and (7.84), starting with the *cubic equation* that is derived from the Lagrangian [i.e., Eq. (7.82)] with a fourth-degree term that is, in the absence of an external field:

$$\begin{aligned} L &= \frac{\hbar c}{2}\left(\overline{\psi}\gamma_\mu[\partial_\mu]\psi \pm l^2\rho^2\right) \quad (l = \text{Const.}) \\ &= \frac{\hbar c}{2i}\left(\xi^+\frac{1}{c}[\partial_t]\xi - \xi^+\mathbf{s}.[\nabla]\xi + \eta^+\frac{1}{c}[\partial_t]\eta + \eta^+\mathbf{s}.[\nabla]\eta \pm il^2\rho^2\right). \end{aligned} \tag{7.85}$$

Bearing in mind the definitions of ω_1 and ω_2, this leads to several equivalent forms of the corresponding equation:

$$\gamma_\mu\partial_\mu\psi m l^2\left(\overline{\psi}\gamma_5\gamma_\mu\psi\right)\gamma_5\gamma_\mu\psi = 0 \tag{7.86a}$$

$$\gamma_\mu\partial_\mu\psi \pm l^2\left(\overline{\psi}\gamma_\mu\psi\right)\gamma_\mu\psi = 0 \tag{7.86b}$$

$$\gamma_\mu\partial_\mu\psi \pm l^2\left[\overline{\psi}\psi - \left(\overline{\psi}\gamma_5\psi\right)\gamma_5\right]\psi = 0 \tag{7.86c}$$

$$\begin{aligned} &\frac{1}{c}\frac{\partial\xi}{\partial t} - \mathbf{s}.\nabla\xi \pm 2il^2\left(\eta^+\xi\right)\eta = 0 \\ &\frac{1}{c}\frac{\partial\eta}{\partial t} - \mathbf{s}.\nabla\eta \pm 2il^2\left(\xi^+\eta\right)\xi = 0. \end{aligned} \tag{7.86d}$$

The first one appears in Heisenberg, (1953,1954), and the second in Finkelstein, Lelevier, and Ruderman (1951), both in particle physics. Heisenberg's equation was later obtained in Rodichev (1961) in a space with torsion. The first three equations are particular cases of the monopole equation (Lochak 1987a,b, 2007a,b). The last equation is nothing but the Weyl representation of any of the other three.

We now come to our second special case of Eqs. (7.83) and (7.84)—namely, the *homogeneous equation*. Here, we choose a linear (with respect to ρ) mass term known as $m_0(\rho)$, but the equation is still nonlinear in ψ:

$$F(\rho) = \kappa_0\rho; \quad \kappa(\rho) = \kappa_0 = \text{Const.} \tag{7.87}$$

So we can rewrite Eq. (7.84) with $k = k_0$, a constant:

$$\begin{aligned} &\frac{1}{c}\frac{\partial\xi}{\partial t}\xi - \mathbf{s}\cdot\nabla\xi - i\frac{g}{\hbar c}(W + \mathbf{s}\cdot\mathbf{B})\xi + i\kappa_0\sqrt{\frac{\eta+\xi}{\xi+\eta}}\eta = 0 \\ &\frac{1}{c}\frac{\partial\eta}{\partial t} - \mathbf{s}\cdot\nabla\eta + i\frac{g}{\hbar c}(W - \mathbf{s}\cdot\mathbf{B})\eta + i\kappa_0\sqrt{\frac{\xi+\eta}{\eta+\xi}}\xi = 0. \end{aligned} \tag{7.88}$$

This is homogeneous of degree 1, with a corresponding second-degree Lagrangian:

$$L_m = \frac{\hbar c}{i}\left\{\begin{aligned} &\xi^+\left(\frac{1}{2}\frac{1}{c}[\partial t] - \frac{g}{\hbar c}W\right)\xi - \xi^+ s\cdot\left(\frac{1}{2}[\nabla] + \frac{g}{\hbar c}\mathbf{B}\right)\xi \\ &+\eta^+\left(\frac{1}{2}\frac{1}{c}[\partial t] + \frac{g}{\hbar c}W\right)\eta + \eta^+ s\cdot\left(\frac{1}{2}[\nabla] + \frac{g}{\hbar c}\mathbf{B}\right)\eta \\ &+2i\kappa_0\sqrt{(\xi+\eta)(\eta+\xi)} \end{aligned}\right\}.$$

Adding in an electric interaction, the equation reads as

$$\begin{aligned} &\frac{1}{c}\frac{\partial\xi}{\partial t} - \mathbf{s}\cdot\nabla\xi - i\frac{e}{\hbar c}(V + \mathbf{s}\cdot\mathbf{A})\xi + i\kappa_0\sqrt{\frac{\eta+\xi}{\xi+\eta}}\eta = 0 \\ &\frac{1}{c}\frac{\partial\eta}{\partial t} - \mathbf{s}\cdot\nabla\eta + i\frac{e}{\hbar c}(V - \mathbf{s}\cdot\mathbf{A})\eta + i\kappa_0\sqrt{\frac{\xi+\eta}{\eta+\xi}}\xi = 0. \end{aligned} \tag{7.90}$$

This particular case was studied by Daviau and Lochak (1991). The Lagrangian equation vanishes by virtue of the equations of the motion, just as in the linear case, and one gets Planck's law again. Some—but not all—solutions also yield de Broglie's wavelength formula and a correct semiclassical approximation. However, the equation does not account for particle-antiparticle pairs.

In closing, we return to a recurrent theme of this chapter: namely, *symmetries,* which we now study in the nonlinear framework, starting with the general Lagrangian equations [Eqs. (7.75) and (7.82)], with linear parts L_D and L_M. We first obtain Table 7.3, which is derived from Tables 7.1 and 7.2.

One finds that the electromagnetic potentials are now C invariant because they are *external* to the system instead of being *emitted*, as in Table 7.2 where they changed signs. The variance of the linear parts L_D and L_M of the Dirac and monopole Lagrangians are given in Table 7.4.

Taking into account the variances of ω_1 and ω_2 as displayed in Table 7.3, we will find those of the nonlinear terms $F(\omega_1,\omega_2)$. Here are some examples;

Table 7.3

		P	T	C
ψ	$\rightarrow$	$\gamma_4\psi$	$-i\gamma_3\gamma_1\psi^*$	$\gamma_2\psi^*$
ξ	$\rightarrow$	η	$s_2\xi^*$	$-is_2\eta^*$
η	$\rightarrow$	ξ	$s_2\eta^*$	$is_2\xi^*$
X_4	$\rightarrow$	Y_4	X_4	Y_4
$\mathbf{X}$	$\rightarrow$	$-\mathbf{Y}$	$-\mathbf{X}$	$\mathbf{Y}$
Y_4	$\rightarrow$	X_4	Y_4	X_4
$\mathbf{Y}$	$\rightarrow$	$-\mathbf{X}$	$-\mathbf{Y}$	$\mathbf{X}$
J_4	$\rightarrow$	J_4	J_4	J_4
$\mathbf{J}$	$\rightarrow$	$-\mathbf{J}$	$-\mathbf{J}$	$\mathbf{J}$
Σ_4	$\rightarrow$	$-\Sigma_4$	Σ_4	$-\Sigma_4$
$\boldsymbol{\Sigma}$	$\rightarrow$	$\boldsymbol{\Sigma}$	$-\boldsymbol{\Sigma}$	$-\boldsymbol{\Sigma}$
ω_1	$\rightarrow$	ω_1	ω_1	$-\omega_1$
ω_2	$\rightarrow$	$-\omega_2$	$-\omega_2$	$-\omega_2$
e	$\rightarrow$	e	$-e$	$-e$
g	$\rightarrow$	g	g	g
V	$\rightarrow$	V	$-V$	V
$\mathbf{A}$	$\rightarrow$	$-\mathbf{A}$	$\mathbf{A}$	$\mathbf{A}$
W	$\rightarrow$	$-W$	W	W
$\mathbf{B}$	$\rightarrow$	$\mathbf{B}$	$-\mathbf{B}$	$\mathbf{B}$

Table 7.4

		P	T	C
L_D	$\rightarrow$	L_D	L_D	$-L_D$
L_M	$\rightarrow$	L_M	L_M	$-L_M$

the variances of the corresponding equations are obtained by combining the information from Tables 7.4 and 7.5.

From Table 7.5, one finds that the three invariances C, P, and T are rarely simultaneously respected; for this to happen, the nonlinear term must be P and T invariant and change sign under C, as do ω_1 and the differential part of the Lagrangian (see Table 7.3). In particular, Heisenberg's equation and that of the magnetic monopole are not C invariant because this invariance is actually *incompatible* with chiral invariance, for the somewhat paradoxical reason that this last invariance entails too much symmetry. Indeed, the mass term of the nonlinear Lagrangian expression is chiral invariant, as a function of the radius ρ, and is thus P, T, and C invariant. For the *equation* to be C invariant, however, one would need the mass term to change sign in the C transform, as previously mentioned. In that sense, this mass term can indeed be called too symmetric. But the P and T

Table 7.5

		P	T	C
ω_1	$\rightarrow$	ω_1	ω_1	$-\omega_1$
ω_2	$\rightarrow$	$-\omega_2$	$-\omega_2$	$-\omega_2$
$F = \omega_1$ (Dirac)	$\rightarrow$	$+F$	$+F$	$-F$
$F = \omega_1^2$	$\rightarrow$	$+F$	$+F$	$+F$
$F = \omega_1^3$	$\rightarrow$	$+F$	$+F$	$-F$
$F = G(\omega_1, \omega_2)\,\omega_1, (G > 0)$	$\rightarrow$	$+F$	$+F$	$-F$
$F = F(\omega_1^2 + \omega_2^2)$	$\rightarrow$	$+F$	$+F$	$+F$
$F = F(\omega_1^2 + \omega_2^2)$ $= -(i\overline{\psi}\gamma_5\gamma_\mu\psi)(i\overline{\psi}\gamma_5\gamma_\mu\psi)$	$\rightarrow$	$+F$	$+F$	$+F$
$F = (\omega_1, \omega_2)$	$\rightarrow$	$-F$	$-F$	$+F$

invariances are still verified, with the definition of T as in Table 7.3. As a result, these chiral-invariant equations do *not* obey the CPT theorem.

The fact that an equation is not C invariant does not by itself preclude the existence of solutions with both signs for the energy, but these do not describe particle-antiparticle pairs as they do in the Dirac equation. Curiously enough, such pairs could exist because of T invariance, with one element going forward in time and the other backward. But the pairs defined via charge conjugation and time inversion, respectively, are not of the same nature. In the first case, the two elements have opposite chiralities and move in the same direction with respect to the course of time, whereas in the second case, both elements have the same chirality but move in opposite time directions (see Table 7.3).

Finally, we would like to conclude by stating that nonlinearity is a complicated matter (an assertion that few would dispute). More information on this subject can be found in Lochak (1997a,b).

CHAPTER 8

A Catalytic Nuclear Fusion Arising from Weak Interaction

8.1 MAIN IDEAS

In this chapter, we shall examine a possible way to avoid the super-high temperatures that are generally introduced in nuclear fusion experiments in order to overcome the Coulomb barrier between electrically charged particles of the same sign. We start from the observation that although temperatures are very high in the middle of stars, they are lower than the temperatures used in terrestrial experiments in nuclear fusion. We suggest the possibility of a catalyst that could be present in stars and absent from our experiments. It will be shown that neutrinos in the stars could play the role of this catalyst since they are abundant and subject to weak interactions, which makes them able to play this role. But they cannot do so in terrestrial experiments, first because there are too few neutrinos, and second because since they have no charge, they are scattered in all directions. Nevertheless, instead of neutrinos, we have at our disposal leptonic magnetic monopoles, which were at first theoretically discovered and described and have now for many years been experimentally produced, observed, and applied. These leptonic monopoles have the same weak interaction properties as neutrinos and could accelerate such phenomena as a proton-proton fusion at temperatures that are far lower than the temperatures used in other attempts at nuclear fusion. In addition, they can easily be focused and accelerated thanks to their magnetic charge, rather than being scattered in space. Thus, they are potentially able to modify the problem of nuclear fusion.

8.2 INTRODUCTION

One of the most difficult problems in applied physics today is the industrial development of nuclear fusion energy. As is well known, the current attempts in this area are based on a race to achieve giant temperatures of plasma (several hundreds of million degrees) in order to increase the velocity of nucleons with the same charge sign (positive or negative), to the extent that they are brought close enough to each other to overcome Coulomb repulsion. But there are problems with this approach.

First, there is no material enclosure that can survive such temperatures. The tokamak, a device invented by Russian physicists Igor Tamm and Andrei Sakharov, seemed to provide an answer: in it, electrically charged particles are forced to rotate around a magnetic field to prevent them from approaching the walls. But the tokamak becomes unstable and works only over brief intervals of time; thus, the problem is not solved.

The message of this chapter is that high temperatures must be abandoned. But how can we do this? Let us start with a few opening remarks:

1. Astrophysicists have discovered that although temperatures inside stars are very high, they are lower than those used in the terrestrial research installations for nuclear fusion. So it is natural to ask whether there is a catalyst in the stars that boosts fusion, but which would be absent from terrestrial laboratories.
2. What could such a catalyst be? It is interesting to observe that weak interactions constitute, in some manner, an obstacle to the strong interactions that provide the energy of stars. For instance, they are in a position to slow down the carbon and hydrogen cycles. Thus, one can ask if, conversely, they could speed up the strong interactions, and whether we could use this property that results from this.
3. The weak interactions that occur in the known astronomical cycles lead to the emission of neutrinos. Conversely, many antineutrinos in the stars are produced by the β disintegration of free neutrons:

$$n \rightarrow p + e^- + \tilde{\nu}. \quad (1)$$

Consequently, these antineutrinos can be absorbed in inverse β disintegration:

$$p + \tilde{\nu} \rightarrow n + e^+. \quad (2)$$

In this last reaction, an antineutrino coming from outside gives rise to the reaction in Eq. (2): its absorption is equivalent to the forced emission of a neutrino. Such a reaction can boost an entire cycle, the antineutrino playing the role of a quasi-catalyst (which is only "quasi" because it disappears in the reaction).

4. Unfortunately, even if this hypothesis is justified, it cannot be applied in laboratories, which do not have sufficient neutrinos. Furthermore, since neutrinos are neutral, they are diffused in all directions and cannot be focused. This is one of the reasons why the famous 1996 experiment of Reines and Cowan, which proved the existence of the neutrino, was very difficult.
5. Nevertheless, a result found by Ivoilov (2006) lends support to this hypothesis; he showed that when irradiating a β radioactive source with leptonic magnetic monopoles, the lifetime of the source decreases; in other words, the β radioactivity is accelerated.

8.3 A POSSIBLE CATALYST FOR NUCLEAR FUSION

Now, we shall show that a possible catalyst for nuclear fusion may be the leptonic nuclear monopoles that were theoretically predicted and then observed in our group (see Lochak, 1983, 1985, 1997a,b; Lochak, 2007; and this entire book). For now, let us say briefly that these monopoles are very light (even massless in the present theory), contrary to the monopoles described by other authors, which are supposed to be very heavy. Furthermore, they have two main properties:

- They have a *magnetic charge* g which is equal to 137/2 times the electron charge (in the same Gaussian units).
- They are magnetically excited neutrinos that are subject to the same weak interactions, and are thus able to influence the same nuclear phenomena. The most important observation is that unlike neutrinos, leptonic monopoles can be focused by electromagnetic forces and their energy can be increased in magnetic fields.

Therefore, we can substitute the following reaction to the reaction in Eq. (2):

$$p + \tilde{m} \rightarrow n + e^{+}, \tag{3}$$

where $\tilde{m}$ is an anti-monopole with the same weak interactions as the antineutrino in Eq. (2). Next we suggest a test experiment in order to verify the reaction (3).

8.3.1 Some Remarks

- *Concerning the reaction in Eq. (3)*: It may be objected that energy is not conserved because a proton is lighter than a neutron. But that was already the case for the β inverse formula [Eq. (2)]: such formulas are only written in conformity with quantum rules, and the conservation of energy is admitted for other reasons. Our case is simpler because the monopole may be accelerated in a magnetic field.
- *Concerning temperature:* The temperature required must be enough to create a plasma, which is much lower than the several hundred million degrees needed for other experiments in nuclear fusion.

8.4 A TEST-EXPERIMENT

Leptonic magnetic monopoles generally appear in disruptive electric phenomena in water, such as the following[31]:

1. An explosion in water of a stepped-up electric conductor (Urutskoiev, Moscow, described at a conference in Nantes, France)
2. An electric arc in water (Ivoilov, Kazan)
3. Strong electric sparks in water (Bergher, Fondation Louis de Broglie, Paris)

The monopoles are recorded in different manners, but most often on photographic film. Here are three characteristic tracks: (a) The first is greatly enlarged (characteristic caterpillar form). (b) The second is a track with its image in a germanium mirror; this image is identical to the original rotated by an angle of π, in accordance with the theoretical predictions. (c) The third track was observed on the magnetic north pole; a solar monopole created by a β decay in a strong solar magnetic field. We have hundreds of other different examples of tracks.

By definition, the recorded monopoles were originally present in the source. Thus, these same monopoles were able to accelerate the creation of deuterons, which is the first and principal stage of the hydrogen cycle:

$$p + p \rightarrow {}^{2}D + e^{+} + \nu. \tag{4}$$

But at this point, there occurs a forced reaction due to an antimonopole coming from outside, which then disappears in the reaction, according to the following formula:

$$p + p + \tilde{m} \rightarrow {}^{2}D + e^{+}. \tag{5}$$

A possible experiment could be to create, in a container of water, the explosion of an electric conductor (as in Urutskoiev's experiments), or an electric arc (as in Ivoilov's experiments). Then, the proportion of heavy water must increase as the water cools in the container.

I am aware that I am making several hypotheses here, but there is no science without hypotheses. Nevertheless, apart from other possible objections, one important question concerns the number of produced monopoles, and thus the probability of the purported nuclear phenomena.

[31] There are other examples, such as β radioactivity in a magnetic field.

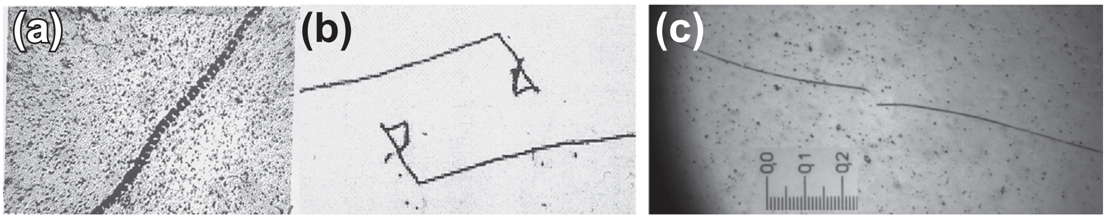

Figure 8.1

Despite that, at first glance, Eq. (5) seems to be easy to realize. However, there is a problem: the number of monopoles (and consequently, the probability of the reaction). If the number of monopoles produced by a source corresponded to the number of trajectories observed in the previous images, this number would be very small and the probability would be negligible.

But an important observation by Ivoilov, confirmed more recently by Daniel and Sonia Fargue (Daviau et al., 2013a,b), shows that apart from the rare large tracks that are generally observed, there are many other tracks which, contrary to the preceding statements, come directly from the direction of the source. These tracks are very thin and are easy to miss. Nevertheless, they have the charge of a monopole, and they probably constitute the hidden but most significant emission of monopoles.

The first to observe these thin tracks (indirectly) was Urutskoiev. When he obtained the first photographs of large monopole tracks on photographic film (curiously, in a plane perpendicular to the direction of the monopole source), he tried to find three-dimensional (3D) tracks in a bubble chamber. But he was disappointed because instead of 3D tracks, he obtained a large white cloud. I now believe that this cloud consisted of the extremely numerous thin tracks later identified by Ivoilov. Actually, the large tracks correspond to deviated monopoles that strike the limit between the plastic film support and the photosensitive coating, and are strongly deviated from their initial trajectory. The large tracks are thus due to a kind of rare accidents, which explains why they are so rare. On the contrary, "true" tracks (which is called "true" because they are free) seem to be the numerous thin tracks. Therefore, we can assert that the emitted monopoles are actually extremely numerous. Thus, it is possible that my test-experiment might be not so difficult to perform. Some other arguments support this idea, among which the macroscopic phenomena observed by Urutskoiev's group after the Chernobyl catastrophe[32], such as the following:

1. In the same hall, there were two parallel conductors separated by some meters: one was a water pipe coming from the nuclear reactor, and the other an electric cable in a concrete box. During the disaster, this concrete protection was broken by the strength of the electric cable strongly attracted by the water pipe. Puzzled by this phenomenon, a young physicist raised the audacious hypothesis that there could be magnetic monopoles in the water coming from the reactor. He meant it as a joke, but actually, it was later confirmed; this implies an enormous quantity of monopoles.
2. The reactor had a concrete cover weighing about 3,000 tons. During the accident, it was pushed aside and fell vertically against the wall of the reactor. If this was due to gas pressure inside the reactor, then a calculation proves that the walls would have exploded, whereas in fact the walls of the reactor were in perfect condition. The paint on the inside of these walls was not even burned, and it would have burned at a temperature of 300°C. Two conclusions emerged from these observations:

   There was no fire in the reactor except in a very limited volume (this was, of course, later verified directly) and thus there was no strong pressure.

   If the cover was lifted, this means that its weight was far less than 3,000 tons during the catastrophe. It was assumed that monopoles have modified gravitation. This was confirmed experimentally by Urutskoiev and theoretically by myself on the basis of my theory of monopoles and of de Broglie's theory of light, which gives a quantum theory of gravitation (de Broglie proved the influence of electromagnetism on gravitation—i.e., Einstein's unitary theory, which he presented in 1942 in his *General Theory of Spin Particles*). All this would be impossible unless an enormous number of monopoles was produced.

[32] It must be emphasized that we absolutely disagree with the official Soviet interpretation of the catastrophe as the result of "errors" by engineers. We have strong arguments in favor of the hypothesis of an explosion of an electrical machine (probably a transformer) that produced a stream of monopoles that then activated a chain of disintegrations never seen before. An important point is that we came up with this hypothesis independently: we met only years later. It must be added that two completely independent groups were sent to Chernobyl to explain the disaster: one by a scientific center, the Kurchatov Institute, under the direction of Urutskoiev, and the other, the official one, in order to find a guilty party. And this is the origin of the official "explanation."

CHAPTER 9

Conclusion

The Foreword of this book presented a brief history of the magnetic monopole. From this, it follows that the monopole is not a particle like others—just another constituent particle of matter. Without its name ever being mentioned, this particular particle was long awaited because it is necessary to the entire concept of electromagnetism. We saw in Chapter 6 that this monopole even plays a role in universal gravitation, in the form of a graviton. If we now take a bird's-eye view of the whole of physics over the last four centuries, we can say that it is dominated by three great ideas: Isaac Newton's universal gravitation, James Clerk Maxwell's electromagnetism, and quantum theory.

And if we want to identify the greatest unachieved dream of the twentieth century, it is most certainly Einstein's Grand Unified Theory, which would have unified the two theories of gravitation and electromagnetism in a single geometric vision of the universe. To this day, such a theory has never been established in the form imagined by its creator. However, although this is little known, Louis de Broglie and Marie-Antoinette Tonnelat[33] gave a different version, based not on geometry but on quantum theory, which emerges from de Broglie's theory of light (see Chapter 6 and de Broglie, 1940–1942, 1949) and his general theory of particles with spin (de Broglie, 1950).

Recall that the idea on which these theories are based is that there exists a fundamental particle of spin ½, defined by Dirac's equation[34], and that the particles of greater spin arise through the fusion of several of these. In particular:

- Two particles of spin ½ give Einstein's photon, of spin 1, which is thus no longer an elementary particle but a composite particle; and de Broglie showed that when the photon is defined this way, it obeys Maxwell's equations. Heisenberg gave as much importance to this discovery of de Broglie's as to the wave properties of the electron (Heisenberg, 1953).

[33] M. A. Tonnelat, an ex-student of de Broglie, was a great specialist in relativity was invited by Schrödinger to Dublin and by Einstein to Princeton.

[34] *Proceedings of the Royal Society, 187,* 610; and *118,* 34; 1928.

- Four particles of spin ½ give the graviton of spin 2, which obeys Einstein's equations of general relativity, unfortunately only in the linear approximation, given the linearity of quantum mechanics. But the most important thing is that the equations of gravitation come together naturally with three of Maxwell's equations of electromagnetism, as in Einstein's vision: it is indeed a version of the Grand Unified Theory, but it is a quantum rather than a geometric theory.
- Furthermore, I recently showed that only two of these photons are Einstein's electric photons (Lochak, 1995a, and Chapter 6 of this volume): the third photon is a magnetic photon that arises in the theory of the monopole. This establishes a link between the magnetic monopole and gravitation.

This book is based on two ideas that form the essence of its content:

- Dirac's electron must be accompanied by another particle, the *leptonic magnetic monopole*.
- Einstein's photon is not unique: it is the fundamental element of a sequence of elecphotons of spins 0, 1, 2, etc… and each electric photon is paired with a magnetic photon.

We already know that the idea of the magnetic monopole—although it was never explicitly named—was expressed by Maxwell and by Pierre Curie. My contribution will be to propose several wave equations, of which one is linear and very simple (1983). At the start of this research, I submitted to Louis de Broglie in his final years (he died in 1987) some preliminary ideas, on the subject of which I will share here a personal anecdote that I have kept to myself until now. One day, he listened to me in silence and then made this single comment: "Too bad Einstein is dead—this would have interested him."

To me, this memory at least partially makes up for the blindness of the scientific community, which is locked into the idea of a very heavy monopole, which has never been observed, and has put up a wall of resistance to my idea, even though it has been repeatedly confirmed by experiments[35]. I can only refer to the saying of Heraclitus: "If you don't seek for the unexpected, you will never know the truth."

Let us now make two remarks on this subject. The idea of the leptonic monopole is hidden in Dirac's equation, in the form of two equations that I

[35] The silence is not absolute, however, since I received from a great institute of nuclear physics the "mathematical proof" that my monopole doesn't exist. The "proof" only had two flaws: (1) It did not start with the correct equation; and (2) It totally neglected the experiments, but that, of course, is a mere detail…

deduced in 1956 thanks to a different formulation of the equation. One of these equations is obvious: it expresses the conservation of electric charge. The other one was incomprehensible at the time: even de Broglie could not decode it, and it took me twenty years to understand that it expresses the conservation of a magnetic quantity which yielded the equation of the monopole.

This first equation leads to a zero mass for the monopole, which must then move in the vacuum at the speed of light, which is not the case for the other two equations found in this book. But it is remarkable that it suffices to write Dirac's usual equation (with a nonzero mass) on the light cone (see Chapter 5), while requiring the current to be isotropic, to render the equation ambivalent, representing both an electron and a monopole.

The second remark concerns de Broglie's theory of light. The original form of the equation that he found is very complicated, but thanks to an algebraic transformation, he was able to deduce (de Broglie, 1950) Maxwell's equation from it with the definition of the fields using potentials and Lorentz's condition. Impressed by this, de Broglie did not look for further algebraic transformations.

It was only recently that I showed that in reality, there are two, and only two, such transformations: de Broglie's and another one (described in Chapter 6). This other transformation also yields Maxwell-like equations, which do not correspond to the electron but to the monopole whose equation I already knew. And they define new covariant derivatives that represent the action of electromagnetism not on an electric charge, but on a magnetic one.

In other words, de Broglie's equations for the photon do not only represent Einstein's photon, but also a second photon of spin 1: namely, a magnetic photon. This second photon introduced by me (and also, independent of me, by Dominique Spehler) is thus not an artificial invention since it was hidden in de Broglie's original equations. It just needed to be found.

It is thus very strange that these two new particles, the magnetic monopole and the magnetic photon, already existed in a hidden form in Dirac's equation for the electron and in de Broglie's equation for light. But I could not show this to de Broglie, who would have understood it in a second and leapt up, interrupting my explanation. Sadly, he had died by then.

Let me add to this what I showed in Chapter 6 concerning the photon of spin 0, which is to my mind a fundamental aspect of de Broglie's theory of light. The application of this to the Aharonov-Bohm effect that I describe shows that this photon of spin 0 represents a kind of closure of the entire theory of light. This circle of ideas constitutes a general theory of electromagnetism, which englobes Einstein's gravitation in an entirely new manner. But it is necessary to state one important criticism: the simple fact—already acknowledged in the theory of de Broglie-Tonnelat—that Einstein's equations of gravitation appear only in the form of linear approximations of general relativity indicates a fundamental incompleteness of the theory.

Unlike the well-known quantum theory, a future theory should start from nonlinear quantum concepts, and the first proof of success would obviously be to find the exact form of general relativity, which should serve as a guide in this research.

As a "conclusion to the conclusion," I would like to add a few further remarks.

The magnetic monopole reveals a duality between electricity and magnetism that Maxwell has already noted in his treaty, since he relied on a double Coulomb's law for electric and magnetic poles. Note that Charles Augustin de Coulomb really established these two laws by measures on electrically charged bodies and at the extremities of long magnetic wires.

We find this duality in what precedes, under two different forms:

- The two different gauges of Dirac's equation, one electric and the other magnetic, which yield the equation of the electron and the equation of the monopole, respectively.
- The second form in which the duality appears is, as we have seen, de Broglie's theory of light and the two photons, electric and magnetic. It must be noted that the magnetic photon already implicitly arose via the second gauge, which gave the equation of the monopole, so that this photon is confirmed by all the experiments on interactions of the monopole with an electromagnetic field, just as Einstein's electric photon is confirmed by the corresponding interactions.

Certainly, there are many less proofs of this duality on the magnetic side than on the electric side, but then they are separated by a century. And it must also be said that when we speak of duality, it is never absolutely symmetric. Our world, at least in the present state of knowledge, is much more electric than magnetic.

This was already visible in wave-particle duality. The material world is much more particle than wave. We know that Heinrich Hertz thought that cathode rays were waves (and his choice is understandable). But Jean Perrin showed the particle properties of these rays, and that characteristic is at the origin of the discovery of the electron. Indeed, de Broglie showed later that both Perrin and Hertz were right about wave-particle duality.

Nevertheless, the electron really is "more particle than wave," whatever Erwin Schrödinger may think. But for the photon, the contrary holds: light is more wave than particle. It is a question of mass—much smaller for the photon than for the electron—and of spin: ½ for the electron, which is an individualistic fermion, and 1 for the photon, which is a boson prone to collective states.

But what happens with a monopole? The equation suggested in Chapters 1, and 3 describes the essential properties of a leptonic monopole. This equation is inspired by the Dirac equation and the change from the electric to the magnetic particle is characterized by the presence of a γ_5 matrix, absent from the Dirac equation. So that the problem of the kinetic moment is very different from the electric case, especially for the Coulomb law.

We saw in Chapter 3 that when a monopole interacts with an electric Coulomb center, we find no more the Laplace functions with integer idexes, as in the case of a hydrogen atom, but generalized spherical functions, the quantum numbers of which are either integers or half-integers. In the case of an integer, we must add the spin value ½: thus, the kinetic moment becomes a half-integer and the monopole is in a fermion state. But if the quantum number of the kinetic moment is a half-integer (which is possible because of the top-symmetry of the monopole) we must, once more, add a spin value ½, so that the total moment will be an integer, and the monopole is now in a boson state: that is a wholly different case.

This is why the interaction between a monopole and an electric charge is a particular case of electromagnetic interaction because, owing to the presence of half-integer kinetic moments of monopoles, due to their top-symmetry, there is the possibility of transitions between fermion and boson states, which is absolutely forbidden. One example of this is between hydrogen quantum states with photon interactions because the latter are bosons. The possible existence of monopoles in boson states implies the possibility of phenomena of the type of magnetic supraconductivity, which could be very interesting.

As was shown in Chapter 3, another important consequence is that in such an electron-monopole interaction, the conservation of the magnetic current is lost because of the quantification of the product (e.g., of both charges, electric and magnetic), and possible jumps between the quantum values. It signifies that, if we admit the general conservation of electricity as seems to be true in all the experiments to date, we must admit that the magnetic charge is not conserved in this special case. It is quantized, and there must be quantum transitions between quantum states. In particular, if the magnetic charge was initially equal to zero, we shall observe the birth of a magnetic monopole. The description of such a phenomenon needs a new equation, which does not seem to be hard to build.

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